FINITE EXTINCTION TIME FOR NON-LINEAR ABSORPTION-DIFFUSION EQUATIONS

THESIS

Jacob B. Goldberg

AFIT/GAM/ENC/02S-02

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

Report Documentation Page

Report Date	Report Type	Dates Covered (from... to)
11 Sep 02	Final	Jun 01 - Sep 02

Title and Subtitle Finite Extinction Time for Non-Linear Absorption-Diffusion Equations	**Contract Number**
	Grant Number
	Program Element Number
Author(s) Mr. Jacob B. Goldberg	**Project Number**
	Task Number
	Work Unit Number
Performing Organization Name(s) and Address(es) Air Force Institute of Technology Graduate School of Engineering and Management (AFIT/EN) 2950 P Street, Bldg 640 WPAFB OH 45433-7765	**Performing Organization Report Number** AFIT/GAM/ENC/02S-02
Sponsoring/Monitoring Agency Name(s) and Address(es)	**Sponsor/Monitor's Acronym(s)**
	Sponsor/Monitor's Report Number(s)

Distribution/Availability Statement
Approved for public release, distribution unlimited

Supplementary Notes

Abstract
In this thesis, we develop a numerical method in order to approximate the solutions of one-dimensional, non-linear absorption-diffusion equations. We test our method for accuracy against a linear diffusion equation with a solution that can be written in closed form. We then test various types of diffusion and absorption terms to determine which ones produce extinction in finite time. We also develop a numerical method to computationally solve diffusion-free equations. We compare the numerical solutions of the one-dimensional, non-linear absorption-diffusion equation and the diffusion-free equation and we find that for the cases tested, the numerical absorption-diffusion solutions are always less than the numerical diffusion-free solutions. Furthermore, we find this is true for the cases tested when there is finite and infinite extinction time. We also look at the open problem where we have slow diffusion and weak absorption but, their combined effect is strong. Our results provide some insight into the answer of this problem.

Subject Terms
Finite Extinction Time, Non-Linear, Absorption-Diffusion, Parabolic

Report Classification unclassified	**Classification of this page** unclassified
Classification of Abstract unclassified	**Limitation of Abstract** UU

Number of Pages
70

AFIT/GAM/ENC/02S-02

FINITE EXTINCTION TIME FOR NON-LINEAR ABSORPTION-DIFFUSION EQUATIONS

THESIS

Jacob B. Goldberg

AFIT/GAM/ENC/02S-02

Approved for public release; distribution unlimited

The views expressed in this thesis are those of the author and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the United States Government.

AFIT/GAM/ENC/02S-02

FINITE EXTINCTION TIME FOR NON-LINEAR ABSORPTION-DIFFUSION EQUATIONS

THESIS

Presented to the Faculty

Department of Mathematics and Statistics

Graduate School of Engineering and Management

Air Force Institute of Technology

Air University

Air Education and Training Command

In Partial Fulfillment of the Requirements for the

Degree of Master of Science in Applied Mathematics

Jacob B. Goldberg, BA

September 2002

Approved for public release; distribution unlimited

AFIT/GAM/ENC/02S-02

FINITE EXTINCTION TIME FOR NON-LINEAR ABSORPTION-DIFFUSION EQUATIONS

Jacob B. Goldberg, BA

Approved:

signed	11 September 2002
Dr. Alan Lair Thesis Advisor	Date
signed	11 September 2002
Dr. Mark Oxley Committee Member	Date
signed	11 September 2002
Dr. Dennis Quinn Committee Member	Date

Acknowledgements

I express my gratitude to those who have helped me in my research. My advisor, Dr. Alan Lair, was greatly instrumental in my work. He provided me with guidance and direction, and helped shape my work into its finished form. I also thank the members of my research committee, Dr. Mark Oxley and Dr. Dennis Quinn. Dr. Oxley provided me with helpful books and papers, and Dr. Quinn provided me with the tools of numerical analysis which will serve as a basis for future work.

I also thank Mr. Eric Howe whose advice and help allowed me to write this thesis in LaTeX.

Finally, I thank Ms. Sebrina Pabon, my family and all the friends I have made here at AFIT. They have helped me let off a little steam, when things got too hot.

Jacob B. Goldberg

Table of Contents

List of Figures

List of Tables

Table Page

AFIT/GAM/ENC/02S-02

Abstract

In this thesis, we develop a numerical method in order to approximate the solutions of one-dimensional, non-linear absorption-diffusion equations. We test our method for accuracy against a linear diffusion equation with a solution that can be written in closed form. We then test various types of diffusion and absorption terms to determine which ones produce extinction in finite time.

We also develop a numerical method to computationally solve diffusion-free equations. We compare the numerical solutions of the one-dimensional, non-linear absorption-diffusion equation and the diffusion-free equation and we find that for the cases tested, the numerical absorption-diffusion solutions are always less than the numerical diffusion-free solutions. Furthermore, we find this is true for the cases tested when there is finite and infinite extinction time.

We also look at the open problem where we have slow diffusion and weak absorption but, their combined effect is strong. Our results provide some insight into the answer of this problem.

FINITE EXTINCTION TIME FOR NON-LINEAR ABSORPTION-DIFFUSION EQUATIONS

I. Introduction

1.1 Motivation

"Diffusion is the process by which matter is transported from one part of a system to another as a result of random molecular motions [4]." Despite the fact that molecules have no preferred direction of motion, there is an overall movement of molecules from regions of higher concentration to regions of lower concentration. This can be explained by the fact that over a given interval of time, on the average, a definite fraction of molecules from the higher concentration will move into the lower concentration, and the same fraction of molecules will move from the lower to higher concentration. So, simply because there are more molecules in the higher concentration, there is a net transfer of molecules from the higher to the lower concentration, as a result of random molecular motions [4]. In addition to concentrations, the matter moving can also be considered as heat flow. Thus, the diffusion equation is commonly referred to as the heat equation.

The n-dimensional, linear diffusion equation is given by

$$u_t(x,t) = \alpha^2 \nabla^2 u(x,t), \qquad x \in \Omega, \qquad 0 < t < \infty, \tag{1.1}$$

where Ω is usually a bounded domain in $\mathbb{R}^n$. A linear equation means that the dependent variable u and all its derivatives appear in a linear fashion, i.e., they are not multiplied together or squared or etc. [6]. Equation (1.1) relates the rate of change in the temperature or concentration profile, $u(x,t)$, with respect to time, $u_t = \frac{\partial u}{\partial t}$, and $\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$, which in one-dimension is the concavity of the temperature or concentration profile. $\nabla^2 u$ in essence compares the temperature or concentration at one point to the temperature or concentration at neighboring points [6]. The quantity $\alpha^2 > 0$ is the diffusion coefficient measured in length^2/time.

When dealing with an internal heat source, an internal heat sink, or an irreversible reaction, $f(x,t,u(x,t))$, the material is either being supplied with heat, heat is being removed, or a reaction is taking place at the location x and instant of time t, and may depend on the temperature or concentration profile, $u(x,t)$. We call f the absorption term. This leads to a modification of the n-dimensional, linear diffusion equation as follows.

$$u_t(x,t) = \alpha^2 \nabla^2 u(x,t) - f(x,t,u(x,t)), \qquad x \in \Omega, \qquad 0 < t < \infty, \tag{1.2}$$

where all the variables are the same as in equation (1.1). Equation (1.2) is often called the absorption-diffusion equation.

Throughout the remainder of this thesis, we will consider a non-linear absorption-diffusion equation, which has the form

$$u_t = \nabla^2 \Phi(u) - F(u), \qquad x \in \Omega, \qquad 0 < t < \infty. \tag{1.3}$$

That is, the absorption term depends only on the temperature or concentration. We will derive the one-dimensional case of equation (1.3) in the next section.

1.2 Derivation

We will consider a one-dimensional, non-linear absorption-diffusion equation with the form

$$u_t(x,t) = \Phi(u(x,t))_{xx} - F(u(x,t)), \qquad 0 < x < L, \qquad 0 < t < \infty. \tag{1.4}$$

We will formally derive equation (1.4) from a heat flow standpoint. From this vantage, conservation of energy is our cornerstone.

Let us suppose we have a thin rod of length L lying centered along the x-axis from 0 to L. We assume the rod is thin so that the temperature at all points of a cross section, A, are constant. This essentially means that we have a one-dimensional rod. Also, let us assume that the rod is laterally insulated so that heat can only flow in the x-direction.

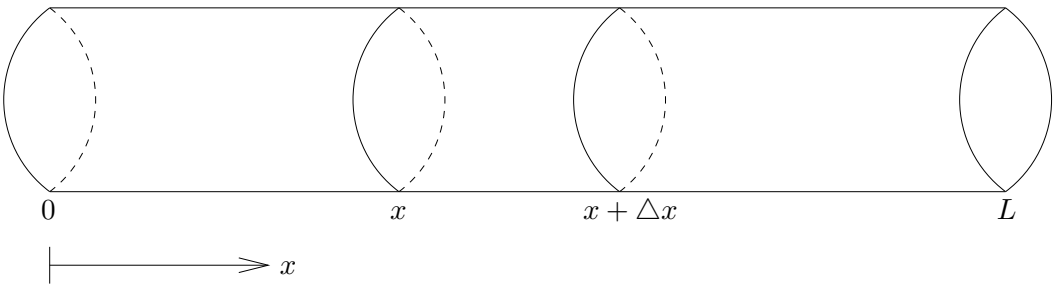

Figure 1.1. Thin Conducting Rod.

Dividing the rod into small segments of length $\triangle x$, we can apply the principle of conservation of heat to a segment of the rod, $[x, x + \triangle x]$. Therefore,

$$\begin{aligned} &\text{Net change of heat inside the segment } [x, x + \triangle x] \\ &= \text{Net change of heat across the boundaries at } x \text{ and } x + \triangle x \\ &+ \text{Total heat generated or absorbed inside the segment } [x, x + \triangle x]. \end{aligned} \tag{1.5}$$

Now, at any time t,

$$\text{Total heat inside the segment } [x, x + \triangle x] = \int_x^{x+\triangle x} Ac(u)\rho(u)u(s,t)ds, \tag{1.6}$$

where $c(u)$ is the thermal capacity of the rod and $\rho(u)$ is the density of the rod, both of which we assume depend on the temperature. Thermal capacity is a material property of the rod which measures its ability to hold heat.

This makes the change of heat inside the segment over time equal to the derivative with respect to t of equation (1.6) as shown below.

$$\begin{aligned} &\text{Net change of heat inside the segment } [x, x + \triangle x] \\ &= \frac{\partial}{\partial t} \int_x^{x+\triangle x} Ac(u)\rho(u)u(s,t)ds = A \int_x^{x+\triangle x} \frac{\partial}{\partial t}[c(u)\rho(u)u(s,t)]ds. \end{aligned} \tag{1.7}$$

Next, we need to find the change in heat across the boundaries at x and $x + \triangle x$.

$$\begin{aligned} &\text{Net change of heat across the boundaries at } x \text{ and } x + \triangle x \\ &= A\{k(u(x + \triangle x, t))u_x(x + \triangle x, t) - k(u(x,t))u_x(x,t)\}, \end{aligned} \tag{1.8}$$

where $k(u)$ is the thermal conductivity of the rod and $u_x(x,t)$ is the thermal gradient at the boundaries at x and $x + \triangle x$. Thermal conductivity is the ability of the material to allow heat to diffuse, and in our case, is dependent on the temperature of the material.

Finally, we must deal with the heat generated or absorbed inside the segment.

$$\begin{aligned} &\text{Total heat generated or absorbed inside the segment } [x, x+\triangle x] \\ &= A\int_x^{x+\triangle x} f(u(s,t))ds, \end{aligned} \tag{1.9}$$

where $f(u(s,t))$ is a function of how the material will generate or absorb heat.

Substituting equations (1.7), (1.8), and (1.9) into equation (1.5) we get

$$\begin{aligned} A\int_x^{x+\triangle x} \frac{\partial}{\partial t}[c(u)\rho(u)u(s,t)]ds &= A[k(u(x+\triangle x,t))u_x(x+\triangle x,t) \\ &\quad - k(u(x,t))u_x(x,t)] + A\int_x^{x+\triangle x} f(u(s,t))ds. \end{aligned} \tag{1.10}$$

At this point, we will assume that $f(u(s,t))$ is a heat sink, i.e., a source of heat absorption in the segment, instead of a heat generator. Thus, $f(u(s,t)) \leq 0$. However, we will change the sign in front of $f(u(s,t))$ so that f itself is nonnegative.

Dividing each side of equation (1.10) by A, and applying the mean value theorem for integrals, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}[c(u(\xi,t))\rho(u(\xi,t))u(\xi,t)]\triangle x &= k(u(x+\triangle x,t))u_x(x+\triangle x,t) \\ &\quad - k(u(x,t))u_x(x,t) - f(u(\xi,t))\triangle x, \end{aligned} \tag{1.11}$$

for some $x < \xi < x + \triangle x$.

Now, dividing both sides of equation (1.11) by $\triangle x$ gives us

$$\begin{aligned} \frac{\partial}{\partial t}[c(u(\xi,t))\rho(u(\xi,t))u(\xi,t)] &= \frac{k(u(x+\triangle x,t))u_x(x+\triangle x,t) - k(u(x,t))u_x(x,t)}{\triangle x} \\ &\quad - f(u(\xi,t)). \end{aligned} \tag{1.12}$$

Letting $\triangle x \to 0$ in equation (1.12) results in

$$\frac{\partial}{\partial t}[c(u(x,t))\rho(u(x,t))u(x,t)] = [k(u(x,t))u_x(x,t)]_x - f(u(x,t)) \tag{1.13}$$

or

$$\frac{\partial}{\partial t}[c(u)\rho(u)u] = [k(u)u_x]_x - f(u). \tag{1.14}$$

Next, we let $g(u) = \int_0^u k(z)dz$ so that $g(u)_{xx} = [k(u)u_x]_x$. This gives us

$$\frac{\partial}{\partial t}[c(u)\rho(u)u] = g(u)_{xx} - f(u). \tag{1.15}$$

Now, we let $w = c(u)\rho(u)u$ and we assume that this relationship is invertible so that we can solve for u to get $u = h(w)$ for some function h. Substituting this into equation (1.15) we obtain

$$\frac{\partial}{\partial t}w = g(h(w))_{xx} - f(h(w)). \tag{1.16}$$

Letting $\Phi(w) = g(h(w))$ and $F(w) = f(h(w))$ in (1.16) we arrive at

$$w_t = \Phi(w)_{xx} - F(w), \tag{1.17}$$

our one-dimensional, non-linear absorption-diffusion equation.

Finally, we switch back to using u instead of w, and note that the rod lies in the interval $[0, L]$ to get $0 < x < L$. This gives

$$u_t = \Phi(u)_{xx} - F(u), \qquad 0 < x < L, \qquad 0 < t < \infty. \tag{1.18}$$

The analogous problem in n-dimensions is given by

$$u_t = \nabla^2\Phi(u) - F(u) \qquad \text{in } \Omega \times (0, \infty), \tag{1.19}$$

where Ω is a bounded domain in $\mathbb{R}^n$.

1.3 Problem Statement

Our goal for this thesis is to determine the types of diffusion terms, Φ, and absorption terms, F, for equation (1.19), which produce extinction in finite time. We would also like to determine which conditions on Φ and F do not produce extinction in finite time.

Definition 1.3.1 *Finite extinction time is a finite time* $t_0 > 0$ *such that* $u(x,t) = 0$ *for all* $(x,t) \in \Omega \times [t_0, \infty)$.

Throughout the remainder of this thesis, we will assume that Φ and F are nondecreasing, nonnegative $C^1((0,\infty)) \cap C([0,\infty))$ functions that satisfy

$$\Phi(0) = 0, \qquad F(0) = 0, \qquad \Phi(s) > 0, \qquad F(s) > 0 \qquad \text{if } s > 0. \tag{1.20}$$

A $C([0,\infty))$ function is a real-valued function that is continuous on the interval $[0,\infty)$. A $C^1((0,\infty))$ function is a real-valued function whose first derivative is continuous on the interval $(0,\infty)$.

The assumptions in (1.20) have physical significance. $\Phi(0) = 0$ means that when there is no heat, i.e., temperature $= 0$, there is no diffusion. $F(0) = 0$ means that there is no absorption if there is no heat. $\Phi(s) > 0$ and $F(s) > 0$ for $s > 0$ means that diffusion and absorption occur, respectively, when there is heat.

We note here that extinction always occurs as $t \to \infty$. However, for some Φ and F, extinction occurs for $t < \infty$. We will prove the former in Section 1.4.

In order to properly consider equation (1.19) so that we may determine extinction times for various Φ and F, we must first set up our boundary and initial conditions. For our boundary condition, we will choose the homogeneous Dirichlet condition,

$$u = 0 \qquad \text{on } \partial\Omega \times [0,\infty). \tag{1.21}$$

This means that on the boundary of Ω, $\partial\Omega$, we will fix the temperature to zero for all time $t \geq 0$. We could have fixed the boundary to be some nonzero constant temperature, or we may also have chosen the temperature at the boundary to change over time, $u = g(t)$, for some nonnegative function g.

Another possible choice for the boundary condition is a Neumann condition, $\frac{\partial u}{\partial \nu} = g(t)$, where $\frac{\partial u}{\partial \nu}$ is the derivative of u in the outward normal direction. This boundary condition specifies heat flow across the boundary or the heat flux. We could have also chosen a Robin boundary condition, $\frac{\partial u}{\partial \nu} + \lambda u = g(t)$, in which there is an interchange of heat between the edge of the material and the surrounding medium. Here λ is some constant.

Now, we must choose our initial condition. We will choose

$$u(x,0) = u_0(x) \geq 0 \qquad \text{on } \bar{\Omega}, \tag{1.22}$$

where $\bar{\Omega} = \Omega \cup \partial\Omega$. That is, u_0 is the initial temperature of the material.

Combining equations (1.19), (1.21), and (1.22), we get an initial-boundary-value problem (IBVP),

$$\begin{aligned} u_t &= \nabla^2\Phi(u) - F(u) && \text{in } \Omega \times (0,\infty), \\ u &= 0 && \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) &= u_0(x) \geq 0 && \text{on } \bar{\Omega}. \end{aligned} \tag{1.23}$$

As noted earlier, we are interested in the extinction time for solutions to equation (1.23). However, it is well-known that, in general, there are no classical solutions to this non-linear parabolic equation for arbitrary choice of Φ and F. In Chapter III, we will derive a numerical algorithm to calculate the solution of the one-dimensional version of equation (1.23). In Chapter IV, we will numerically calculate the solution for several choices of Φ and F in order to determine whether finite extinction time occurs.

Now, let us look at a related problem given by the diffusion-free equation,

$$\begin{aligned} z'(t) &= -[\Phi(z(t)) + F(z(t))], \qquad 0 < t < \infty, \\ z(0) &= M > 0. \end{aligned} \tag{1.24}$$

It can be shown that the solution, $z(t)$, to equation (1.24), has finite extinction time if and only if

$$\int_0^{\epsilon} \frac{ds}{\Phi(s) + F(s)} < \infty, \tag{1.25}$$

where $\epsilon > 0$.

We want to determine if u will have a finite extinction time, when Φ and F satisfy condition (1.25). So, if we have a Φ and F that satisfy condition (1.25), then z will have a finite extinction time. If we then compare the nonnegative solutions u and z, and find that $u(x,t) \leq z(t)$ for all x and t, then u will also have a finite extinction time. It can be shown analytically that $u \leq z$ for the case where equation (1.24) has no Φ term. That is, if z satisfies

$$\begin{aligned} z'(t) &= -F(z), \qquad 0 < t < \infty, \\ z(0) &= M, \end{aligned} \tag{1.26}$$

and $u(x,0) \leq M$ where u is a solution of (1.23), then $u \leq z$. It has not been shown for equation (1.24).

We should also mention that if (1.25) holds, and we can calculate

$$\int_0^{z} \frac{ds}{\Phi(s) + F(s)}, \tag{1.27}$$

then the unique solution of equation (1.24) is z where $z(t)$ satisfies

$$\int_M^{z(t)} \frac{ds}{\Phi(s) + F(s)} = -\int_0^t dt = -t, \tag{1.28}$$

and no numerical approximation is needed.

The numerical results for equation (1.24) as well as a comparison to the solution of the one-dimensional version of (1.23) are given in Chapter IV. The derivation of the numerical solution for (1.24) is given in Chapter III.

1.4 Proof that extinction always occurs as $t \to \infty$

Theorem 1.4.1 *Suppose u is a nonnegative solution of equation (1.23), where Φ and F are nondecreasing, nonnegative $C^1((0,\infty)) \cap C([0,\infty))$ functions that satisfy (1.20). Then $\int_\Omega \int_0^{u(x,t)} F(s)dsdx \to 0$ as $t \to \infty$.*

Proof. First, we multiply equation (1.19) by $F(u)$ and we integrate over Ω to get

$$\int_\Omega F(u)u_t dx = \int_\Omega F(u)\nabla^2\Phi(u)dx - \int_\Omega [F(u)]^2 dx. \tag{1.29}$$

Consider the term $\int_\Omega F(u)\nabla^2\Phi(u)dx$. We see that with integration by parts,

$$\int_\Omega F(u)\nabla^2\Phi(u)dx = \int_\Omega \nabla[F(u)\nabla\Phi(u)] - \int_\Omega \nabla F(u)\nabla\Phi(u)dx. \tag{1.30}$$

Now, we apply the divergence theorem to equation (1.30) and obtain

$$\begin{aligned} \int_\Omega F(u)\nabla^2\Phi(u)dx &= \int_{\partial\Omega} F(u)\nabla\Phi(u)\cdot\vec{\nu}ds - \int_\Omega \nabla F(u)\cdot\nabla\Phi(u)dx \\ &= -\int_\Omega \nabla F(u)\cdot\nabla\Phi(u)dx, \end{aligned} \tag{1.31}$$

where $\vec{\nu}$ is the unit outward normal to $\partial\Omega$, ds denotes an element of surface area, and $\int_{\partial\Omega} F(u)\nabla\Phi(u)\cdot\vec{\nu}ds = 0$ because u vanishes on $\partial\Omega$ and F(0)=0.

Now, substituting (1.31) back into equation (1.29) we arrive at

$$\begin{aligned} \int_\Omega F(u)u_t dx &= -\int_\Omega \nabla F(u)\cdot\nabla\Phi(u)dx - \int_\Omega [F(u)]^2 dx \\ &= -\int_\Omega F'(u)\Phi'(u)|\nabla u|^2 dx - \int_\Omega [F(u)]^2 dx \\ &\le -\int_\Omega [F(u)]^2 dx, \end{aligned} \tag{1.32}$$

because $F(u)$ and $\Phi(u)$ are nondecreasing.

Next, we define $G(u) = \int_0^u F(s)ds$ so that equation (1.32) becomes

$$\frac{\partial}{\partial t}\int_\Omega G(u(x,t))dx \le -\int_\Omega [F(u(x,t))]^2 dx. \tag{1.33}$$

However, since $F'(u) \geq 0$, then

$$G(u) = \int_0^u F(s)ds \leq \int_0^u F(u)ds = uF(u). \tag{1.34}$$

Taking the integral of both sides of equation (1.34), we get

$$\int_\Omega G(u)dx \leq \int_\Omega uF(u) \leq \sqrt{\int_\Omega u^2 dx}\sqrt{\int_\Omega [F(u)]^2 dx}. \tag{1.35}$$

On a side note, if we start with equation (1.19), and multiply both sides by u, then integrate by parts and apply the divergence theorem as we did above, we obtain

$$\begin{aligned}
\int_\Omega uu_t dx &= \int_\Omega u\nabla^2\Phi(u)dx - \int_\Omega uF(u)dx \\
&= -\int_\Omega \nabla u \cdot \nabla\Phi(u)dx - \int_\Omega uF(u)dx \\
&= -\int_\Omega \nabla u \cdot \Phi'(u)\nabla u - \int_\Omega uF(u) \\
&= -\int_\Omega \Phi'(u)|\nabla u|^2 - \int_\Omega uF(u) \\
&\leq 0,
\end{aligned} \tag{1.36}$$

because $\Phi'(u) \geq 0$, $F(u)$ nonnegative, and $u > 0$.

We see that

$$\frac{\partial}{\partial t}\int_\Omega u^2 dx = 2\int_\Omega uu_t dx, \tag{1.37}$$

and therefore,

$$\frac{\partial}{\partial t}\int_\Omega u^2 dx \leq 0. \tag{1.38}$$

Equation (1.38) means that $\int_\Omega u^2(x,t)dx$ is bounded above, or in other words,

$$\int_\Omega u^2(x,t)dx \leq \int_\Omega u_0^2(x)dx \equiv M^2, \tag{1.39}$$

where $u_0(x)$ is the initial condition, and M^2 is a constant.

Now, substituting (1.39) into equation (1.35) we obtain

$$\int_\Omega G(u)dx \leq M\sqrt{\int_\Omega [F(u)]^2 dx}. \tag{1.40}$$

Squaring both sides of equation (1.40) and dividing by $-M^2$ gives

$$-\frac{1}{M^2}\left(\int_\Omega G(u)dx\right)^2 \geq -\int_\Omega [F(u)]^2 dx. \tag{1.41}$$

Next, we combine equations (1.33) and (1.41) to get

$$\frac{\partial}{\partial t}\int_\Omega G(u)dx \leq -\int_\Omega [F(u)]^2 dx \leq -\frac{1}{M^2}\left(\int_\Omega G(u)dx\right)^2. \tag{1.42}$$

Let $w(t) = \int_\Omega G(u(x,t))dx$, so that the inequality in (1.42) becomes

$$w'(t) \leq -\frac{1}{M^2}w^2(t). \tag{1.43}$$

Divide both sides by $w^2(t)$ and recognize that $\frac{w'(t)}{w^2(t)} = -\frac{d}{dt}\left(\frac{1}{w(t)}\right)$ to arrive at

$$-\frac{d}{dt}\left(\frac{1}{w(t)}\right) \leq -\frac{1}{M^2}. \tag{1.44}$$

Next, we integrate both sides with respect to t to get

$$-\frac{1}{w(t)} + \frac{1}{w(0)} \leq -\frac{t}{M^2}. \tag{1.45}$$

Finally, we solve for $w(t)$ to obtain

$$0 \leq w(t) \leq \frac{1}{\frac{1}{w(0)} + \frac{t}{M^2}}, \tag{1.46}$$

which approaches zero as $t \to \infty$. Thus, extinction always occurs as $t \to \infty$. Q.E.D.

In summary, we know that all nonnegative solutions of (1.23) have extinction time, though it may be at infinity. We wish to determine whether extinction occurs in finite or infinite time for any given Φ and F. We will try to do this directly by numerically

calculating the solution of the one-dimensional version of equation (1.23). We will also try to do this indirectly by attempting to show numerically that any solution u of the one-dimensional version of (1.23) must satisfy $u \leq z$ where z is the solution to (1.24) with $u(x, 0) \leq M$.

II. Literature Review

In this thesis, we are studying a computational model of non-linear variables Φ and F, that predicts temperatures using finite-difference approximations. In predicting temperatures, we need to consider specific boundary and initial conditions. With all of this at hand, we hope to be able to determine whether finite extinction time occurs.

In this chapter, we will cite a few examples of related problems. Among these are heat pipe problems, dead core reaction-diffusion problems, and non-linear parabolic finite extinction time problems.

2.1 Heat Pipe Problems

Heat pipes have been studied for many years. Among these studies is the transient behavior of heat pipes under many external conditions. Under normal conditions, it has been determined that transient response is caused by the thermal capacity and conductance of the shell, capillary structure, and working fluid, and is only slightly influenced by liquid and vapor dynamics [3].

Chang and Colwell [3] study low-temperature, heat pipe operation. They develop a computational model for predicting the transient temperatures of low-temperature heat pipes based on finite-difference approximations. They consider variations of thermal properties using cubic spline interpolation and various boundary conditions for thermal coupling. They also calculate nodal temperatures using an alternating direction implicit method.

Chang and Colwell then compare their results to experimental values that they produced in the lab. The final predicted steady-state values are in good agreement with the experimental values. However, the predicted temperatures reach steady-state faster than the measured values mainly because of the assumption of a perfectly insulated heat pipe and uniform vapor temperature [3].

2.2 Dead Core Problems

Dead core problems are very interesting in nature. The problems themselves are brought about by the diffusion of one substance through the pores of a solid body which, with the evolution or absorption of heat, may absorb and immobilize some of the diffusing substance. The heat itself will then diffuse through the medium, affecting the amount of substance the solid can absorb [4]. If a region of zero reactant concentration is formed in finite time, we have a dead core problem [2]. More specifically, a dead core is a region, $\Omega_0 \subseteq \Omega$, that is formed in a finite time t_0 such that $u(x,t) = 0$ for all $(x,t) \in \Omega_0 \times [t_0, \infty)$.

Bandle and Stakgold [2] study non-linear reaction-diffusion problems in which a dead core may be formed. They formulate the conditions needed for a dead core to exist, and calculate estimates for its time of onset. The simple model Bandle and Stakgold produce is a system of homogeneous non-linear parabolic equations with boundary and initial conditions. The main parameters of which are both concentration and temperature.

Bandle and Stakgold [2] show that if there is strong reaction, there will be a dead core formation. They also derive estimates for the time of onset, size, and location of the dead core. Finally, they prove convexity of the dead core.

2.3 Finite Extinction Time Problems

In this thesis, we are concerned with the non-linear parabolic finite extinction time problem whose solution, u, is always zero on the boundary, and whose initial function, u_0, is nonnegative. Therefore, we will consider the problem

$$\begin{aligned} u_t &= \triangle\varphi(u) - f(u) && \text{in } \Omega \times (0,\infty), \\ u &= 0 && \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) &= u_0(x) \geq 0 && \text{on } \bar{\Omega}, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, and φ and f are nondecreasing, nonnegative $C^1((0,\infty)) \cap C([0,\infty))$ functions that satisfy

$$\varphi(0) = 0, \qquad f(0) \geq 0, \qquad \varphi(s) > 0, \qquad f(s) > 0 \qquad \text{if } s > 0. \tag{2.2}$$

Kalashnikov [7] and Kersner [8] both show that the nonnegative solution of the one-dimensional version of (2.1) with $\Omega = \mathbb{R}$, $\varphi'' \geq 0$, and $\varphi'(0) < \infty$, has finite extinction time if the condition of strong absorption,

$$\int_0^\epsilon \frac{ds}{f(s)} < \infty, \tag{2.3}$$

holds. Conversely, Kalashnikov [7] shows, among other things, that if (2.3) is infinite and the condition $\varphi'(s)f(s) \leq Ks$ for some constant $K \geq 0$ holds, then the solution does not have a finite extinction time.

Diaz and Diaz [5] consider equation (2.1) with $f = 0$ and show that if Ω is bounded, then a necessary and sufficient condition for u to have finite extinction time is the condition of fast diffusion, i.e.,

$$\int_0^\epsilon \frac{ds}{\varphi(s)} < \infty. \tag{2.4}$$

Lair [9] shows that if either (2.3) or (2.4) holds, then the solution to (2.1) has finite extinction time. Lair and Oxley [10] extend Lair's result to show that if

$$\int_0^\epsilon \frac{ds}{\varphi(s) + f(s)} = \infty \tag{2.5}$$

holds then there is no finite extinction time. Equivalently, they also show that if finite extinction time exists, then the integral in (2.5) is finite.

The above results lead to an open, unsolved problem. Suppose f and φ satisfy

$$\int_0^\epsilon \frac{ds}{f(s)} = \infty, \qquad \int_0^\epsilon \frac{ds}{\varphi(s)} = \infty, \qquad \text{and} \qquad \int_0^\epsilon \frac{ds}{\varphi(s) + f(s)} < \infty. \tag{2.6}$$

This is to say that we have slow diffusion and weak absorption, but their combined effect is strong. Will any solution of equation (2.1) have a finite extinction time? This is unknown even for the special case where φ and f are both concave down [10].

The primary objective of this thesis is to provide an answer to this open problem. Or, if unable to answer it directly, to provide computational evidence of the correct answer

and perhaps give insight into how to prove analytically what the computations, in the examples considered, indicate.

III. Numerical Analysis

In this chapter, we will derive our numerical method for the solution of equation (1.23) in one-dimension. We will use a predictor-corrector method in order to ensure the accuracy of our solution. Later in the chapter, we will derive our numerical solution to equation (1.24).

3.1 Corrector for absorption-diffusion problem

In this section, we will develop a numerical method for the solution of the one-dimensional version of equation (1.23) with $\Omega = (0, 1)$. We start by choosing a method that mirrors the Crank-Nicolson method because when this method is used to solve linear partial differential equations, it converges and is numerically stable for all $\triangle x > 0$ and $\triangle t > 0$. It also has a global accuracy of $\mathcal{O}(\triangle x^2)$ [12]. Later, we will see that we need another numerical method to act as a predictor to help solve this method. We should note here that this implicit method can be made into a system of linear equations, with one equation for each x node. This system can then be written in a matrix-vector form with a tri-diagonal matrix, so that the system is easily solvable.

Starting with equation (1.18), we see that we can rewrite this in terms of partial derivatives.

$$\begin{aligned} u_t &= [\Phi(u)]_{xx} - F(u) \\ &= \left[\frac{\partial}{\partial u}\Phi(u)\frac{\partial u}{\partial x}\right]_x - F(u) \\ &= \frac{\partial}{\partial x}[\Phi'(u)u_x] - F(u) \\ &= \Phi'(u)u_{xx} + \Phi''(u)(u_x)^2 - F(u). \end{aligned} \tag{3.1}$$

Rewriting the equation this way, allows us to create a finite difference scheme. We could have also created a finite difference scheme directly from equation (1.18), where

$$\Phi(u)_{xx} = \frac{\Phi(u(x + \triangle x, t)) - 2\Phi(u(x, t)) + \Phi(u(x - \triangle x, t))}{(\triangle x)^2}, \tag{3.2}$$

but this method was found to be unstable for $\Phi(u) = u^p$ with $0 < p < 1$.

First, we will discretize in space. Starting with the Taylor expansion of the solution, $u(x,t)$, a small distance, $\triangle x$, away from x, we have

$$u(x+\triangle x,t) = u(x,t) + \triangle x\, u_x(x,t) + \frac{\triangle x^2}{2}u_{xx}(x,t) + \frac{\triangle x^3}{6}u_{xxx}(x,t) + \cdots \tag{3.3}$$

and

$$u(x-\triangle x,t) = u(x,t) - \triangle x\, u_x(x,t) + \frac{\triangle x^2}{2}u_{xx}(x,t) - \frac{\triangle x^3}{6}u_{xxx}(x,t) + \cdots\ , \tag{3.4}$$

where $\triangle x^n = (\triangle x)^n$. Subtracting (3.4) from (3.3) and truncating the series, we arrive at

$$u(x+\triangle x,t) - u(x-\triangle x,t) = 2\triangle x\, u_x(x,t) + \frac{\triangle x^3}{3}u_{xxx}(\xi,t), \tag{3.5}$$

where $x-\triangle x \le \xi \le x+\triangle x$.

Rearranging (3.5) in terms of u_x we get

$$u_x(x,t) = \frac{u(x+\triangle x,t) - u(x-\triangle x,t)}{2\triangle x} - \frac{\triangle x^2}{6}u_{xxx}(\xi,t). \tag{3.6}$$

Now, in our particular problem, our space is the closed interval $[0,1]$. If we divide this space into N equal intervals of length $\triangle x$, then we have nodes, $x_i = i\triangle x$, where $i = 0,1,\ldots,N$. Therefore, x_0 is on the boundary at 0 and x_N is at 1. We let $u_i(t) = u(x_i,t)$ in equation (3.6) and we regard $-\frac{\triangle x^2}{6}u_{xxx}(\xi,t)$ as the error term in our approximation of u_x. We note that the error on the approximation of u_x is $\mathcal{O}(\triangle x^2)$. This gives us an approximation for u_x at each node,

$$u_x(x_i,t) \approx \frac{u_{i+1}(t) - u_{i-1}(t)}{2\triangle x}. \tag{3.7}$$

Going back, if we add equations (3.3) and (3.4), truncate, and make the proper substitutions as described above, we arrive at an approximation for u_{xx} at each node x_i,

$$u_{xx}(x_i,t) \approx \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\triangle x^2}. \tag{3.8}$$

We note here that the error on the approximation is $-\frac{\triangle x^2}{12}u_{xxxx}(\xi,t)$ which is $\mathcal{O}(\triangle x^2)$.

If we substitute our newly calculated approximations, (3.7) and (3.8), into equation (3.1), we arrive at a spatially discretized version of our problem,

$$\begin{aligned}\frac{\partial}{\partial t}u_i(t) &= \Phi'(u_i(t))\left[\frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{\triangle x^2}\right] \\ &\quad + \Phi''(u_i(t))\left[\frac{u_{i+1}(t)-u_{i-1}(t)}{2\triangle x}\right]^2 - F(u_i(t)).\end{aligned} \tag{3.9}$$

At this point we see that we have derivatives of Φ in (3.9). However, since we know Φ, we can *a priori* calculate its derivatives, and substitute them in where necessary. This seems to be a problem for cases where Φ' and Φ'' are undefined on the boundary where $u = 0$, but it is not because we never need to solve for a solution directly on any boundary. This is because our boundary conditions force our solution to be zero on the boundaries.

We still have a time derivative left in equation (3.9), and therefore, we need to discretize in time. First, we will take the integral of both sides of equation (3.9) over a small interval of time from t to $t + \triangle t$.

$$\begin{aligned}\int_t^{t+\triangle t} u_t(x_i,t)dt &= \int_t^{t+\triangle t}\left\{\Phi'(u_i(t))\left[\frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{\triangle x^2}\right]\right. \\ &\quad \left. + \Phi''(u_i(t))\left[\frac{u_{i+1}(t)-u_{i-1}(t)}{2\triangle x}\right]^2 - F(u_i(t))\right\}dt.\end{aligned} \tag{3.10}$$

Notice that $\int_t^{t+\triangle t} u_t(x_i,t)dt = u_i(t+\triangle t) - u_i(t)$. Now, with the remaining integral we use the trapezoid rule and we arrange the equation so that only $u_i(t+\triangle t)$ is on the left hand side to arrive at

$$\begin{aligned}u_i(t+\triangle t) &= u_i(t) + \frac{\triangle t}{2}\left\{\Phi'(u_i(t+\triangle t))\left[\frac{u_{i+1}(t+\triangle t)-2u_i(t+\triangle t)+u_{i-1}(t+\triangle t)}{\triangle x^2}\right]\right. \\ &\quad + \Phi''(u_i(t+\triangle t))\left[\frac{u_{i+1}(t+\triangle t)-u_{i-1}(t+\triangle t)}{2\triangle x}\right]^2 - F(u_i(t+\triangle t)) \\ &\quad + \Phi'(u_i(t))\left[\frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{\triangle x^2}\right] \\ &\quad \left. + \Phi''(u_i(t))\left[\frac{u_{i+1}(t)-u_{i-1}(t)}{2\triangle x}\right]^2 - F(u_i(t))\right\}.\end{aligned} \tag{3.11}$$

Here we stop to note that the trapezoid rule on a linear equation has a local interpolation error of $\mathcal{O}(\triangle t^3)$ [1]. This means that our use of the trapezoid rule produces a minimum local error of $\mathcal{O}(\triangle t^3)$ because in our case the error also depends on Φ, F, and our $\mathcal{O}(\triangle x^2)$ approximations. Also, if $\Phi = u$, $|\frac{\partial F}{\partial u}|$ is small, and $\triangle t$ is small, then this method will converge [12]. It is our assumption that this method will converge for any Φ or F.

In our problem, the time interval is $[0, \infty)$. Dividing this interval up into subintervals of length $\triangle t$, we arrive at nodes, $t_n = n\triangle t$, where $n = 0, 1, \ldots$. This means t_0 is at the initial time, 0. We will now let $u_{i\,n} = u_i(t_n) = u(x_i, t_n)$ in equation (3.11) thus giving

$$\begin{aligned} u_{i\,n+1} \;=\; & u_{i\,n} + \frac{\triangle t}{2}\left\{\Phi'(u_{i\,n+1})\left[\frac{u_{i+1\,n+1} - 2u_{i\,n+1} + u_{i-1\,n+1}}{\triangle x^2}\right]\right. \\ & + \Phi''(u_{i\,n+1})\left[\frac{u_{i+1\,n+1} - u_{i-1\,n+1}}{2\triangle x}\right]^2 - F(u_{i\,n+1}) \\ & \left. + \Phi'(u_{i\,n})\left[\frac{u_{i+1\,n} - 2u_{i\,n} + u_{i-1\,n}}{\triangle x^2}\right] + \Phi''(u_{i\,n})\left[\frac{u_{i+1\,n} - u_{i-1\,n}}{2\triangle x}\right]^2 - F(u_{i\,n})\right\}. \end{aligned} \tag{3.12}$$

Multiplying out the squared terms and reorganizing the terms in a manner similar to the arrangement of terms in the Crank-Nicolson method, we obtain

$$\begin{aligned} & u_{i\,n+1}\left[1 + \frac{\triangle t}{\triangle x^2}\Phi'(u_{i\,n+1})\right] + u_{i+1\,n+1}\left[-\frac{\triangle t}{2\triangle x^2}\Phi'(u_{i\,n+1}) - \frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n+1})u_{i+1\,n+1}\right. \\ & \left. + \frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n+1})u_{i-1\,n+1}\right] + u_{i-1\,n+1}\left[-\frac{\triangle t}{2\triangle x^2}\Phi'(u_{i\,n+1}) - \frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n+1})u_{i-1\,n+1}\right. \\ & \left. + \frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n+1})u_{i+1\,n+1}\right] = u_{i\,n}\left[1 - \frac{\triangle t}{\triangle x^2}\Phi'(u_{i\,n})\right] + u_{i+1\,n}\left[\frac{\triangle t}{2\triangle x^2}\Phi'(u_{i\,n})\right. \\ & \left. + \frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n})u_{i+1\,n} - \frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n})u_{i-1\,n}\right] + u_{i-1\,n}\left[\frac{\triangle t}{2\triangle x^2}\Phi'(u_{i\,n})\right. \\ & \left. + \frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n})u_{i-1\,n} - \frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n})u_{i+1\,n}\right] - \frac{\triangle t}{2}\left[F(u_{i\,n+1}) + F(u_{i\,n})\right]. \end{aligned} \tag{3.13}$$

Now that we have arranged the equation the way we want, we see that we have a numerical scheme for our problem. However, before we solve our problem, we can make our work a little easier by writing equation (3.13) in matrix form. This will allow us to

solve the entire x interval at once for each time step.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_{1\,n+1} & a_{1\,n+1} & b_{1\,n+1} & & 0 & 0 \\ 0 & c_{2\,n+1} & a_{2\,n+1} & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & a_{N-1\,n+1} & b_{N-1\,n+1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{0\,n+1} \\ u_{1\,n+1} \\ u_{2\,n+1} \\ \vdots \\ u_{N-1\,n+1} \\ u_{N\,n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ f_{1\,n} & d_{1\,n} & e_{1\,n} & & 0 & 0 \\ 0 & f_{2\,n} & d_{2\,n} & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & d_{N-1\,n} & e_{N-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{0\,n} \\ u_{1\,n} \\ u_{2\,n} \\ \vdots \\ u_{N-1\,n} \\ u_{N\,n} \end{bmatrix} \tag{3.14}$$

$$- \frac{\triangle t}{2} \begin{bmatrix} F(u_{0\,n+1}) + F(u_{0\,n}) \\ F(u_{1\,n+1}) + F(u_{1\,n}) \\ F(u_{2\,n+1}) + F(u_{2\,n}) \\ \vdots \\ F(u_{N-1\,n+1}) + F(u_{N-1\,n}) \\ F(u_{N\,n+1}) + F(u_{N\,n}) \end{bmatrix},$$

where

$$
\begin{aligned}
a_{i\,n+1} &= 1+\frac{\triangle t}{\triangle x^2}\Phi'(u_{i\,n+1}),\\
b_{i\,n+1} &= -\frac{\triangle t}{2\triangle x^2}\Phi'(u_{i\,n+1})-\frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n+1})u_{i+1\,n+1}+\frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n+1})u_{i-1\,n+1},\\
c_{i\,n+1} &= -\frac{\triangle t}{2\triangle x^2}\Phi'(u_{i\,n+1})-\frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n+1})u_{i-1\,n+1}+\frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n+1})u_{i+1\,n+1},\\
d_{i\,n} &= 1-\frac{\triangle t}{\triangle x^2}\Phi'(u_{i\,n}),\\
e_{i\,n} &= \frac{\triangle t}{2\triangle x^2}\Phi'(u_{i\,n})+\frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n})u_{i+1\,n}-\frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n})u_{i-1\,n},\\
f_{i\,n} &= \frac{\triangle t}{2\triangle x^2}\Phi'(u_{i\,n})+\frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n})u_{i-1\,n}-\frac{\triangle t}{8\triangle x^2}\Phi''(u_{i\,n})u_{i+1\,n}.
\end{aligned}
$$

Note that we can incorporate our boundary conditions of $u_{0\,n}=0$ and $u_{N\,n}=0$ into equation (3.14). This will simplify the equation to

$$
\begin{bmatrix}
a_{1\,n+1} & b_{1\,n+1} & 0 & \cdots & 0\\
c_{2\,n+1} & a_{2\,n+1} & b_{2\,n+1} & & 0\\
0 & c_{3\,n+1} & a_{3\,n+1} & & 0\\
\vdots & & & \ddots & \vdots\\
0 & 0 & 0 & \cdots & a_{N-1\,n+1}
\end{bmatrix}
\begin{bmatrix} u_{1\,n+1}\\ u_{2\,n+1}\\ u_{3\,n+1}\\ \vdots\\ u_{N-1\,n+1}\end{bmatrix}
$$

$$
=\begin{bmatrix}
d_{1\,n} & e_{1\,n} & 0 & \cdots & 0\\
f_{2\,n} & d_{2\,n} & e_{2\,n} & & 0\\
0 & f_{3\,n} & d_{3\,n} & & 0\\
\vdots & & & \ddots & \vdots\\
0 & 0 & 0 & \cdots & d_{N-1\,n}
\end{bmatrix}
\begin{bmatrix} u_{1\,n}\\ u_{2\,n}\\ u_{3\,n}\\ \vdots\\ u_{N-1\,n}\end{bmatrix}
-\frac{\triangle t}{2}
\begin{bmatrix} F(u_{1\,n+1})+F(u_{1\,n})\\ F(u_{2\,n+1})+F(u_{2\,n})\\ F(u_{3\,n+1})+F(u_{3\,n})\\ \vdots\\ F(u_{N-1\,n+1})+F(u_{N-1\,n})\end{bmatrix}, \tag{3.15}
$$

where $a_{i\,n+1}$, $b_{i\,n+1}$, $c_{i\,n+1}$, $d_{i\,n}$, $e_{i\,n}$, and $f_{i\,n}$ are the same as in equation (3.14). We should stop to note that this method has a minimum global error of $\mathcal{O}(\triangle x^2)$ because it degenerates into the linear Crank-Nicolson method as will be shown in Section 3.2. The exact error is dependent upon the choice of Φ and F.

Representing the $a_{i\,n+1}$, $b_{i\,n+1}$, $c_{i\,n+1}$ matrix in equation (3.15) by A_{n+1}, the $u_{i\,n+1}$ vector with $\vec{u}_{n+1}$, the $d_{i\,n}$, $e_{i\,n}$, $f_{i\,n}$ matrix with B_n, the $u_{i\,n}$ vector with $\vec{u}_n$, and the $F(u_{i\,n+1}) + F(u_{i\,n})$ vector with $\vec{F}_n$, we obtain

$$A_{n+1}\vec{u}_{n+1} = B_n\vec{u}_n - \frac{\triangle t}{2}\vec{F}_n. \tag{3.16}$$

Finally, we solve for $\vec{u}_{n+1}$ to get

$$\vec{u}_{n+1} = A_{n+1}^{-1}B_n\vec{u}_n - \frac{\triangle t}{2}A_{n+1}^{-1}\vec{F}_n. \tag{3.17}$$

When solving equation (3.17), we run into a problem. Unlike the linear case of the Crank-Nicolson method, we need to estimate the solution at the $n+1$ time step in order to calculate the solution at the $n+1$ time step. This occurs because the non-linearity of our equation adds extra $u_{i\,n}$ and $u_{i\,n+1}$ terms to the tri-diagonal matrices. However, this is not a major problem since we can use equation (3.17) as our corrector in a predictor-corrector method.

We will choose a forward difference method as our predictor because it is a good, standard explicit method with a local error of $\mathcal{O}(\triangle x^2)$ for linear problems [1]. By making a good guess at our solution with the forward difference method, we have assured that our solution has a minimum global error of $\mathcal{O}(\triangle x^2)$. This is particularly important when solving a non-linear problem such as ours. The predictor will be formulated in Section 3.3.

3.2 Check of corrector formulation

Before we move on to the formulation of the predictor, we should first check to see if our method degenerates to the Crank-Nicolson method when our problem becomes linear. We must realize that for the linear, one-dimensional case of equation (1.23), $\Phi(u) = u$. This means that $\Phi'(u) = 1$ and $\Phi''(u) = 0$. Also, the absorption term will only depend upon x and t in the linear case. Thus, $F(u(x,t)) = F(x,t)$. With all this in mind, we shall

substitute these values into (3.15) to give us

$$\begin{bmatrix} 1+\frac{\triangle t}{\triangle x^2} & \frac{-\triangle t}{2\triangle x^2} & 0 & \cdots & 0 \\ \frac{-\triangle t}{2\triangle x^2} & 1+\frac{\triangle t}{\triangle x^2} & \frac{-\triangle t}{2\triangle x^2} & & 0 \\ 0 & \frac{-\triangle t}{2\triangle x^2} & 1+\frac{\triangle t}{\triangle x^2} & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\frac{\triangle t}{\triangle x^2} \end{bmatrix} \begin{bmatrix} u_{1\,n+1} \\ u_{2\,n+1} \\ u_{3\,n+1} \\ \vdots \\ u_{N-1\,n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{\triangle t}{\triangle x^2} & \frac{\triangle t}{2\triangle x^2} & 0 & \cdots & 0 \\ \frac{\triangle t}{2\triangle x^2} & 1-\frac{\triangle t}{\triangle x^2} & \frac{\triangle t}{2\triangle x^2} & & 0 \\ 0 & \frac{\triangle t}{2\triangle x^2} & 1-\frac{\triangle t}{\triangle x^2} & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\frac{\triangle t}{\triangle x^2} \end{bmatrix} \begin{bmatrix} u_{1\,n} \\ u_{2\,n} \\ u_{3\,n} \\ \vdots \\ u_{N-1\,n} \end{bmatrix} \tag{3.18}$$

$$- \frac{\triangle t}{2} \begin{bmatrix} F(x_1,t_{n+1})+F(x_1,t_n) \\ F(x_2,t_{n+1})+F(x_2,t_n) \\ F(x_3,t_{n+1})+F(x_3,t_n) \\ \vdots \\ F(x_{N-1},t_{n+1})+F(x_{N-1},t_n) \end{bmatrix},$$

which is the Crank-Nicolson method for a linear absorption-diffusion equation with zero Dirichlet boundary conditions.

We note here that by degenerating the corrector method to the Crank-Nicolson method, and if $F = 0$, we remove the need for a predictor. This is because there are no longer any $u_{i\,n+1}$ terms in the matrices of equation (3.18). The method, however, is still an implicit method.

3.3 Predictor for absorption-diffusion problem

We have already decided to use a forward difference method as our predictor so we begin as before with equation (1.18). We then follow all of the same steps as in Section 3.1 until we arrive at equation (3.10). Again we note that $\int_t^{t+\triangle t} u_t(x_i,t)dt = u_i(t+\triangle t) - u_i(t)$,

and we make the substitution to give

$$\begin{aligned} u_i(t+\triangle t) - u_i(t) &= \int_t^{t+\triangle t} \left\{ \Phi'(u_i(t)) \left[\frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\triangle x^2} \right] \right. \\ &\quad \left. + \Phi''(u_i(t)) \left[\frac{u_{i+1}(t) - u_{i-1}(t)}{2\triangle x} \right]^2 - F(u_i(t)) \right\} dt. \end{aligned} \tag{3.19}$$

We will now use the forward rectangle rule on the remaining integral in equation (3.19). We will also simultaneously make the substitution $u_{i\,n} = u_i(t_n)$ and leave only the $u_{i\,n+1}$ term on the left hand side. This gives

$$\begin{aligned} u_{i\,n+1} &= u_{i\,n} + \triangle t \left\{ \Phi'(u_{i\,n}) \left[\frac{u_{i+1\,n} - 2u_{i\,n} + u_{i-1\,n}}{\triangle x^2} \right] \right. \\ &\quad \left. + \Phi''(u_{i\,n}) \left[\frac{u_{i+1\,n} - u_{i-1\,n}}{2\triangle x} \right]^2 - F(u_{i\,n}) \right\}. \end{aligned} \tag{3.20}$$

It is from this step that we gain a truncation error which has a minimum error of $\mathcal{O}(\triangle t^2)$. The actual error depends on our choice of Φ and F.

Multiplying out all the terms in (3.20) and rearranging in an appropriate manner, we arrive at

$$\begin{aligned} u_{i\,n+1} &= u_{i\,n} \left[1 - \frac{2\triangle t}{\triangle x^2} \Phi'(u_{i\,n}) \right] + u_{i+1\,n} \left[\frac{\triangle t}{\triangle x^2} \Phi'(u_{i\,n}) + \frac{\triangle t}{4\triangle x^2} \Phi''(u_{i\,n}) u_{i+1\,n} \right. \\ &\quad \left. - \frac{\triangle t}{4\triangle x^2} \Phi''(u_{i\,n}) u_{i-1\,n} \right] + u_{i-1\,n} \left[\frac{\triangle t}{\triangle x^2} \Phi'(u_{i\,n}) + \frac{\triangle t}{4\triangle x^2} \Phi''(u_{i\,n}) u_{i-1\,n} \right. \\ &\quad \left. - \frac{\triangle t}{4\triangle x^2} \Phi''(u_{i\,n}) u_{i+1\,n} \right] - \triangle t F(u_{i\,n}). \end{aligned} \tag{3.21}$$

Now, we have a numerical scheme for our problem, but just as in Section 3.1, we can make our work a little easier by writing equation (3.21) in matrix form. This will allow us to

solve the entire x interval at once for each time step.

$$\begin{bmatrix} u_{0\,n+1} \\ u_{1\,n+1} \\ u_{2\,n+1} \\ \vdots \\ u_{N-1\,n+1} \\ u_{N\,n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ p_{1\,n} & g_{1\,n} & h_{1\,n} & & 0 & 0 \\ 0 & p_{2\,n} & g_{2\,n} & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & g_{N-1\,n} & h_{N-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{0\,n} \\ u_{1\,n} \\ u_{2\,n} \\ \vdots \\ u_{N-1\,n} \\ u_{N\,n} \end{bmatrix} - \triangle t \begin{bmatrix} F(u_{0\,n}) \\ F(u_{1\,n}) \\ F(u_{2\,n}) \\ \vdots \\ F(u_{N-1\,n}) \\ F(u_{N\,n}) \end{bmatrix}, \tag{3.22}$$

where

$$\begin{aligned} g_{i\,n} &= 1 - \frac{2\triangle t}{\triangle x^2}\Phi'(u_{i\,n}), \\ h_{i\,n} &= \frac{\triangle t}{\triangle x^2}\Phi'(u_{i\,n}) + \frac{\triangle t}{4\triangle x^2}\Phi''(u_{i\,n})u_{i+1\,n} - \frac{\triangle t}{4\triangle x^2}\Phi''(u_{i\,n})u_{i-1\,n}, \\ p_{i\,n} &= \frac{\triangle t}{\triangle x^2}\Phi'(u_{i\,n}) + \frac{\triangle t}{4\triangle x^2}\Phi''(u_{i\,n})u_{i-1\,n} - \frac{\triangle t}{4\triangle x^2}\Phi''(u_{i\,n})u_{i+1\,n}. \end{aligned}$$

We can now incorporate our boundary conditions, $u_{0\,n} = 0$ and $u_{N\,n} = 0$, into equation (3.22). This will simplify the equation to

$$\begin{bmatrix} u_{1\,n+1} \\ u_{2\,n+1} \\ u_{3\,n+1} \\ \vdots \\ u_{N-1\,n+1} \end{bmatrix} = \begin{bmatrix} g_{1\,n} & h_{1\,n} & 0 & \cdots & 0 \\ p_{2\,n} & g_{2\,n} & h_{2\,n} & & 0 \\ 0 & p_{3\,n} & g_{3\,n} & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_{N-1\,n} \end{bmatrix} \begin{bmatrix} u_{1\,n} \\ u_{2\,n} \\ u_{3\,n} \\ \vdots \\ u_{N-1\,n} \end{bmatrix} - \triangle t \begin{bmatrix} F(u_{1\,n}) \\ F(u_{2\,n}) \\ F(u_{3\,n}) \\ \vdots \\ F(u_{N-1\,n}) \end{bmatrix}, \tag{3.23}$$

where $g_{i\,n}$, $h_{i\,n}$, and $p_{i\,n}$ are the same as in equation (3.22).

Representing the $u_{i\,n+1}$ vector in equation (3.23) with $\vec{u}_{n+1}^{(p)}$, the $g_{i\,n}$, $h_{i\,n}$, $p_{i\,n}$ matrix with C_n, the $u_{i\,n}$ vector with $\vec{u}_n^{(p)}$, and the $F(u_{i\,n})$ vector with $\vec{F}$, we obtain

$$\vec{u}_{n+1}^{(p)} = C_n \vec{u}_n^{(p)} - \triangle t \vec{F}, \tag{3.24}$$

where the superscript (p) denotes predictor.

We now have enough information to generate an accurate numerical solution to our problem. Before we do, we should note that equation (3.24) has the stability condition that $\frac{\Phi'(u)\triangle t}{\triangle x^2} \leq \frac{1}{2}$. We shall show this in the next section.

3.4 Stability condition for the predictor of the absorption-diffusion problem

In order to show that our predictor has the stability condition $\frac{\Phi'(u)\triangle t}{\triangle x^2} \leq \frac{1}{2}$, we begin with equation (3.1),

$$u_t = \Phi'(u)u_{xx} + \Phi''(u)(u_x)^2 - F(u). \tag{3.25}$$

In general, the absorption term plays no role in stability, so we will neglect this term. Now, we make a first order approximation for $\Phi''(u)(u_x)^2$, which is zero. This leaves us with

$$u_t = \Phi'(u)u_{xx}, \tag{3.26}$$

which contains the major contributors to the predictor methods stability condition.

Discretizing equation (3.26) in the same manner as the predictor method described in Section 3.3 gives

$$u_{i\,n+1} = u_{i\,n} + \triangle t\, \Phi'(u_{i\,n}) \left[\frac{u_{i+1\,n} - 2u_{i\,n} + u_{i-1\,n}}{\triangle x^2} \right]. \tag{3.27}$$

We will now use von Neumann's method. First, we get

$$p^{n+1}e^{ij\theta} = p^n e^{ij\theta} + \frac{\Phi'(u)\triangle t}{\triangle x^2} \left[p^n e^{(i+1)j\theta} - 2p^n e^{ij\theta} + p^n e^{(i-1)j\theta} \right], \tag{3.28}$$

where $j = \sqrt{-1}$ and θ is arbitrary.

We solve for our error growth factor, p, to obtain

$$\begin{aligned} p &= 1 + \frac{\Phi'(u)\triangle t}{\triangle x^2}\left[e^{j\theta} + e^{-j\theta} - 2\right] \\ &= 1 + 2\frac{\Phi'(u)\triangle t}{\triangle x^2}\left[\cos(\theta) - 1\right] \\ &= 1 - 4\frac{\Phi'(u)\triangle t}{\triangle x^2}\sin^2\left(\frac{\theta}{2}\right). \end{aligned} \tag{3.29}$$

Now, we need $|p| < 1$. Therefore,

$$-1 \leq 1 - 4\frac{\Phi'(u)\triangle t}{\triangle x^2}\sin^2\left(\frac{\theta}{2}\right) \leq 1, \tag{3.30}$$

which with some manipulation becomes

$$0 \leq \frac{\Phi'(u)\triangle t}{\triangle x^2}\sin^2\left(\frac{\theta}{2}\right) \leq \frac{1}{2}. \tag{3.31}$$

Since θ is arbitrary,

$$\frac{\Phi'(u)\triangle t}{\triangle x^2} \leq \frac{1}{2}, \tag{3.32}$$

where $\Phi' > 0$.

3.5 Use of predictor-corrector for absorption-diffusion problem

Now, we just need to put it all together. First, we choose a $\triangle x$ and $\triangle t$ that satisfy the stability condition, $\frac{\Phi'(u)\triangle t}{\triangle x^2} \leq \frac{1}{2}$, of the predictor. By doing this, we assure that our predictor method will be computationally stable. This means that if we introduce an error at any time step, the error will not increase as we proceed. Next, we know that our initial condition is $u(x,0) = u_0(x)$, so we create our initial vector, $\vec{u}_0^{(p)}$, from this. We then solve equation (3.24) for $\vec{u}_1^{(p)}$. Now, we substitute $\vec{u}_1^{(p)}$ and $\vec{u}_0^{(p)}$ into A_{n+1}, B_n, $\vec{u}_n$, and $\vec{F}_n$ in equation (3.16). Next, we solve (3.16) for $\vec{u}_1$ as shown in equation (3.17). Then, we let $\vec{u}_1 = \vec{u}_1^{(p)}$ and we use this as our new predictor in equation (3.24). Finally, we repeat the process for each time step until we have reached our final time step. In cases with finite

extinction time, our final time step is at the time of extinction. For cases with infinite extinction time, we allow the computer to run until $u_{in} \leq 10^{-324}$. See Section 4.2 for a more detailed explanation.

3.6 Predictor-corrector formulation and use for diffusion-free problem

Now that we have produced a numerical method which can solve the one-dimensional version of equation (1.23) with zero Dirichlet boundary conditions, we can move on to creating a numerical method for solving equation (1.24). We shall start by taking the integral of both sides of equation (1.24) from time t to $t + \triangle t$ in order to get

$$\int_t^{t+\triangle t} z'(t)dt = -\int_t^{t+\triangle t} [\Phi(z(t)) + F(z(t))]dt. \tag{3.33}$$

We note that $\int_t^{t+\triangle t} z'(t)dt = z(t+\triangle t) - z(t)$ and so we substitute this into equation (3.33) in order to get

$$z(t + \triangle t) - z(t) = -\int_t^{t+\triangle t} [\Phi(z(t)) + F(z(t))]dt. \tag{3.34}$$

We now use the trapezoid rule to evaluate the integral on the right hand side of (3.34). This gives

$$z(t + \triangle t) - z(t) = -\frac{\triangle t}{2}[\Phi(z(t + \triangle t) + F(z(t + \triangle t)) + \Phi(z(t)) + F(z(t))]. \tag{3.35}$$

The truncation error from using the trapezoid rule is at minimum $\mathcal{O}(\triangle t^3)$ and depends on the choice of Φ and F.

Now, we can divide our time interval, $[0, \infty)$, into subintervals of length $\triangle t$ to obtain nodes, $t_n = n\triangle t$, where $n = 0, 1, \ldots$. This means $t_0 = 0$. It also means that we can make our nodes the same distance apart as we have done for our numerical solution of the one-dimensional version of (1.23) just by choosing the same $\triangle t$. This will allow us to more easily compare the solutions of equations (1.23) and (1.24). We let $z_n = z(t_n)$ in equation (3.35) and we separate the n and $n + 1$ terms thus giving

$$z_{n+1} + \frac{\triangle t}{2}[\Phi(z_{n+1}) + F(z_{n+1})] = z_n + \frac{\triangle t}{2}[\Phi(z_n) + F(z_n)]. \tag{3.36}$$

Now, we have an implicit method for the solution of (1.24). This means that we will have to make an estimate for the solution at the $n+1$ time step in order to solve equation (3.36). We will do this with a predictor, and we will make (3.36) our corrector. For our predictor, we will go back to equation (3.34). Instead of using a trapezoid method to estimate the integral on the right hand side of the equation, we will use a forward difference method. By doing this we get

$$z(t+\triangle t) - z(t) = -\triangle t[\Phi(z(t)) + F(z(t))]. \tag{3.37}$$

This step adds a minimum truncation error of $\mathcal{O}(\triangle t^2)$ depending on Φ and F.

Again, we make our $z_n = z(t_n)$ substitution and solve for z_{n+1} to give us

$$z_{n+1} = z_n - \triangle t[\Phi(z_n) + F(z_n)]. \tag{3.38}$$

As in Section 3.3, we now change the notation slightly so that we can follow the same procedure as in Section 3.5 to numerically solve equation (1.24) with a predictor-corrector method. So, equation (3.38) is now

$$z_{n+1}^{(p)} = z_n^{(p)} - \triangle t[\Phi(z_n^{(p)}) + F(z_n^{(p)})]. \tag{3.39}$$

It occurs to us at this point, we could have just used the predictor method as our sole solution to equation (1.24). Using just the predictor would give us a minimum local error of $\mathcal{O}(\triangle t)$ which is accurate, but when dealing with a non-linear problem such as ours, we wish to be as accurate as possible. So, we use the predictor-corrector method to increase our accuracy.

IV. Numerical Results

In this chapter, we will see that our numerical method for the solution of the one-dimensional version of equation (1.23) is an accurate solution when the problem is linear. We will also see that our method can show that for certain cases of Φ and F, the solution has finite extinction time, consistent with the analytical results of Lair [9]. At the end of the chapter, we will compare the solutions of equation (1.23) and equation (1.24).

4.1 Accuracy of the numerical solution of the absorption-diffusion problem

We know that our numerical method for the one-dimensional version of equation (1.23) degenerates to the Crank-Nicolson method for linear problems. We also know that for linear problems, the Crank-Nicolson method has an $\mathcal{O}(\triangle x^2)$ global accuracy. This means that we can test a linear problem with a known analytic solution and compare it with our numerical results.

First, we will choose an equation with a known solution. Let us start with a simple diffusion equation problem with zero boundary conditions and an initial function $u_0(x) = \sin(\pi x)$. So, we shall solve

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t), \qquad 0 < x < 1, \qquad 0 < t < \infty, \\ u(0,t) &= 0, \qquad\qquad\qquad\quad\;\; 0 < t < \infty, \\ u(1,t) &= 0, \qquad\qquad\qquad\quad\;\; 0 < t < \infty, \\ u(x,0) &= \sin(\pi x), \qquad 0 \le x \le 1. \end{aligned} \tag{4.1}$$

We know that this has an exact solution of $u_e(x,t) = u(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

Now that we have chosen a problem with an exact solution, we will now choose $\triangle x$ and $\triangle t$. We will start by choosing multiple values of $\triangle x$. This way, we will be able to compare each solution to the exact solution. This will allow us to see if we achieve the correct order of accuracy. We will choose $\triangle x$ equal to 0.1, 0.05, and 0.02.

Even though we do not need to satisfy the stability requirement for the predictor because this a linear problem with $F = 0$ (see Section 3.2), we will anyway. So, we must choose $\triangle t$ to satisfy the condition $\frac{\triangle t}{\triangle x^2} \le \frac{1}{2}$ because $\Phi'(u) = 1$. Therefore, $\triangle t$ must be less

than or equal to 0.005, 0.00125, and 0.0002, respectively. We will choose $\triangle t = 0.001$ for $\triangle x = 0.1$ and 0.05, and we will choose $\triangle t = 0.0002$ for $\triangle x = 0.02$.

We will make calculations at $t = 0.5$, 1, and 5. For ease of use and lack of space, we only show the solutions at spatial increments of $\triangle x = 0.1$. We will define our solutions for $\triangle x$ equal to 0.1, 0.05, and 0.02 to be $u_{.1}(x,t)$, $u_{.05}(x,t)$, and $u_{.02}(x,t)$, respectively. Also, we will define our errors to be $e_{.1}(x,t) = u_{.1}(x,t) - u_e(x,t)$, $e_{.05}(x,t) = u_{.05}(x,t) - u_e(x,t)$, and $e_{.02}(x,t) = u_{.02}(x,t) - u_e(x,t)$.

Table 4.1. Numeric and exact results of equation (4.1) at $t = 0.5$. For $\triangle x = 0.1$ and 0.5, $\triangle t = 0.001$. For $\triangle x = 0.02$, $\triangle t = 0.0002$. All values are multiplied by 10^{-3}.

	$u_{.1}$	$u_{.05}$	$u_{.02}$	u_e	$e_{.1}$	$e_{.05}$	$e_{.02}$
$u(.1,.5)$	2.314072	2.244971	2.226021	2.222414	0.091658	0.022557	0.003607
$u(.2,.5)$	4.401627	4.270189	4.234144	4.227283	0.174344	0.042906	0.006861
$u(.3,.5)$	6.058320	5.877411	5.827799	5.818356	0.239964	0.059055	0.009443
$u(.4,.5)$	7.121982	6.909311	6.850989	6.839888	0.282094	0.069423	0.011101
$u(.5,.5)$	7.488495	7.264879	7.203556	7.191883	0.296612	0.072996	0.011673
$u(.6,.5)$	7.121982	6.909311	6.850989	6.839888	0.282094	0.069423	0.011101
$u(.7,.5)$	6.058320	5.877411	5.827799	5.818356	0.239964	0.059055	0.009443
$u(.8,.5)$	4.401627	4.270189	4.234144	4.227283	0.174344	0.042906	0.006861
$u(.9,.5)$	2.314072	2.244971	2.226021	2.222414	0.091658	0.022557	0.003607

We see from Table 4.1, that at $t = 0.5$, we improve our accuracy as we lower our $\triangle x$, just as we should. If we average the errors for each node of a given $\triangle x$, we get the average error for the numerical solution with that $\triangle x$. With $\triangle x = 0.1$, our average error is 2.08081×10^{-4}. For $\triangle x = 0.05$ we get an average error of 5.12087×10^{-5}, and we get an average error of 8.18856×10^{-6} for $\triangle x = 0.02$.

In going from $\triangle x = 0.1$ to $\triangle x = 0.05$, we double the number of nodes. If our method is actually $\mathcal{O}(\triangle x^2)$, then our error should be reduced by a factor of 4. If we divide our average error at $\triangle x = 0.1$ by 4, we get 5.20203×10^{-5}, which is approximately equal to our average error with $\triangle x = 0.05$. Therefore, we see that our numerical method for equation (4.1) has a global error of $\mathcal{O}(\triangle x^2)$.

We can also look at the relative error. Relative error is defined as $\frac{e_i}{u_e}$, where $i = 0.1$, 0.05, and 0.02. If we average the relative errors of the nodes for each $\triangle x$, we get the

average relative error for that $\triangle x$. Our average relative errors at $t = 0.5$ are 0.04124, 0.01015, and 0.00162, for $\triangle x = 0.1$, 0.05, and 0.02, respectively.

Table 4.2. Numeric and exact results of equation (4.1) at $t = 1$. For $\triangle x = 0.1$ and 0.5, $\triangle t = 0.001$. For $\triangle x = 0.02$, $\triangle t = 0.0002$. All values are multiplied by 10^{-5}.

	$u_{.1}$	$u_{.05}$	$u_{.02}$	u_e	$e_{.1}$	$e_{.05}$	$e_{.02}$
$u(.1,1)$	1.732892	1.630945	1.603527	1.598334	0.134558	0.032611	0.005193
$u(.2,1)$	3.296156	3.102241	3.050089	3.040213	0.255943	0.062028	0.009876
$u(.3,1)$	4.536770	4.269868	4.198087	4.184494	0.352276	0.085374	0.013593
$u(.4,1)$	5.333292	5.019531	4.935148	4.919167	0.414125	0.100364	0.015981
$u(.5,1)$	5.607756	5.277847	5.189121	5.172319	0.435437	0.105528	0.016802
$u(.6,1)$	5.333292	5.019531	4.935148	4.919167	0.414125	0.100364	0.015981
$u(.7,1)$	4.536770	4.269868	4.198087	4.184494	0.352276	0.085374	0.013593
$u(.8,1)$	3.296156	3.102241	3.050089	3.040213	0.255943	0.062028	0.009876
$u(.9,1)$	1.732892	1.630945	1.603527	1.598334	0.134558	0.032611	0.005193

Calculating the average errors at $t = 1$, we arrive at 3.05471×10^{-6}, 7.40313×10^{-7}, and 1.17876×10^{-7} for $\triangle x = 0.1$, 0.05, and 0.02, respectively. Again we see that our global error is of $\mathcal{O}(\triangle x^2)$. Now we calculate the average relative errors to obtain 0.08419, 0.02040, and 0.00325, respectively.

Table 4.3. Numeric and exact results of equation (4.1) at $t = 5$. For $\triangle x = 0.1$ and 0.5, $\triangle t = 0.001$. For $\triangle x = 0.02$, $\triangle t = 0.0002$. All values are multiplied by 10^{-22}.

	$u_{.1}$	$u_{.05}$	$u_{.02}$	u_e	$e_{.1}$	$e_{.05}$	$e_{.02}$
$u(.1,5)$	1.713672	1.265513	1.162657	1.143954	0.569718	0.121559	0.018703
$u(.2,5)$	3.259598	2.407148	2.211505	2.175931	1.083667	0.231217	0.035574
$u(.3,5)$	4.486452	3.313155	3.043875	2.994912	1.491540	0.318243	0.048963
$u(.4,5)$	5.274141	3.894847	3.578290	3.520730	1.753411	0.374117	0.057560
$u(.5,5)$	5.545560	4.095285	3.762436	3.701914	1.843646	0.393371	0.060522
$u(.6,5)$	5.274141	3.894847	3.578290	3.520730	1.753411	0.374117	0.057560
$u(.7,5)$	4.486452	3.313155	3.043875	2.994912	1.491540	0.318243	0.048963
$u(.8,5)$	3.259598	2.407148	2.211505	2.175931	1.083667	0.231217	0.035574
$u(.9,5)$	1.713672	1.265513	1.162657	1.143954	0.569718	0.121559	0.018703

The average errors at $t = 5$ are 1.29337×10^{-22}, 2.75960×10^{-23}, and 4.24580×10^{-24}, respectively. The average relative errors are 0.49802, 0.10626, and 0.01635, respectively. We notice that our relative errors seem to grow in an almost linear relation to t. This means our solution will eventually become relatively inaccurate. At $t = 5$ with $\triangle x = 0.1$,

we are already starting to see a loss of relative accuracy. So, it is important to choose $\triangle x$ and $\triangle t$ wisely.

Since this is the simplest one-dimensional, linear case of the diffusion equation, our numerical method will generate the minimum amount of error possible. So, thinking ahead, we may believe we could have accuracy problems for large t, especially for poorly chosen $\triangle x$ and $\triangle t$. Since, in general, there are no classical solutions to (1.23), we cannot test the accuracy of our method for non-linear equations. The only test that we can do is to try various Φ and F to see if finite extinction time occurs, and that these results are consistent with those of Lair [9]. This is done in the next section.

4.2 Finite extinction time of solutions

Now that we know our numerical solution is accurate, at least for linear equations, we need to see if it can correctly predict finite extinction time. To do this, we will use a modification of Lair's [9] result. We consider the equation

$$u_t = [u^p]_{xx} - u^q, \tag{4.2}$$

for some $p > 0$ and $q \geq 0$. We know that the solution of (4.2) has finite extinction time if and only if $\min\{p, q\} < 1$. For all of the calculations in this section we assume the zero Dirichlet boundary condition and the initial condition,

$$\begin{aligned} u(0,t) &= 0, & 0 < t < \infty, \\ u(1,t) &= 0, & 0 < t < \infty, \\ u(x,0) &= \sin(\pi x), & 0 \leq x \leq 1. \end{aligned} \tag{4.3}$$

First, we will test the case where $p = \frac{1}{2}$ and $q = 1$. This corresponds to $\Phi = \sqrt{u}$ and $F = u$. This case should have finite extinction time, and we see from Figure 4.1 that the solution has an extinction time at $t = 0.205$.

This is a good time to stop and note that the code used to calculate the above solution, and all of the other solutions in this chapter, has a command built-in that will

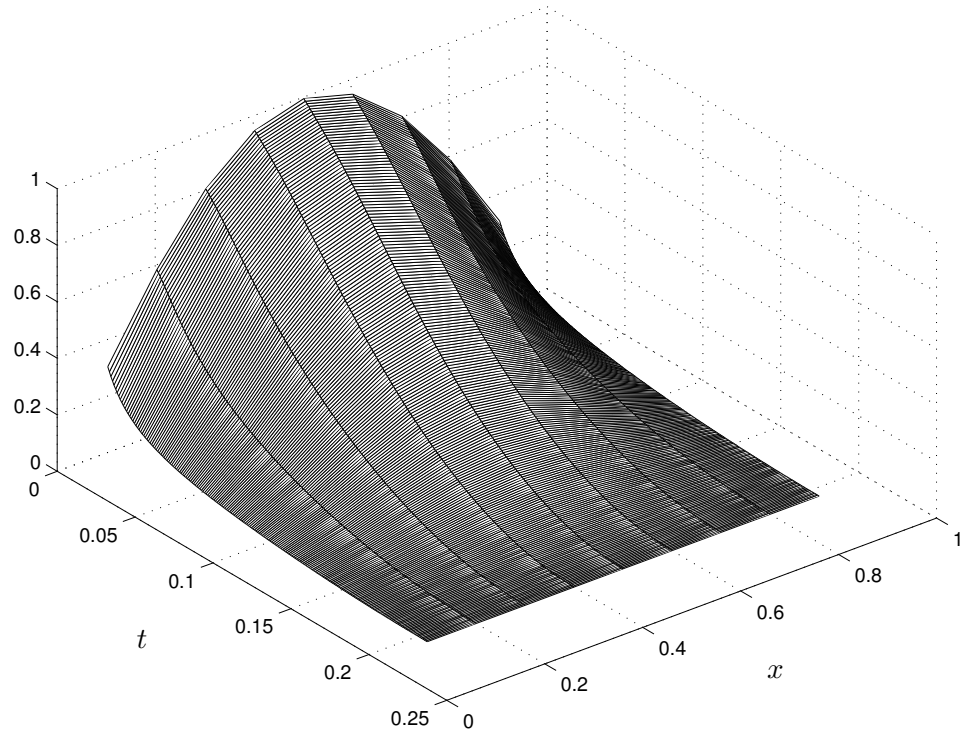

Figure 4.1. Solution to equation (1.23) with $\Phi = \sqrt{u}$ and $F = u$. $\triangle x = 0.1$ and $\triangle t = 0.001$.

not allow the calculated solution to become negative. If the calculated solution at any node, for any iteration, becomes negative, the code will set the calculated solution at that node to zero. Therefore, once the calculated solution at any node reaches zero, it cannot go any lower. It can, however, go back up again from zero, but this was never observed. The justification for this command is that we are only interested in nonnegative solutions to our equation.

We shall also note that this is a case where Φ' and Φ'' are undefined at $x = 0$, in fact, $\lim_{s\to 0} \Phi'(s) = \infty$ and $\lim_{s\to 0} \Phi''(s) = \infty$. We choose $\triangle x = 0.1$ and $\triangle t = 0.001$ because they satisfy the stability condition $\frac{\Phi'(u)\triangle t}{\triangle x^2} \leq \frac{1}{2}$ at t=0, and the computer is able to handle the amount of processing needed to calculate the numerical solution under these conditions. With $\triangle x = 0.05$, we need to choose a $\triangle t$ so small that the computer cannot handle the processing needed to calculate the numerical solution.

For the next few examples, we will choose different combinations of Φ and F so that our solutions will have finite extinction times. We shall choose $\Phi = \sqrt{u}$ and $F = u^3$ for our second example. This particular solution reaches extinction at $t = 0.212$ as shown in Figure 4.2. Again we note here that we have chosen $\triangle x = 0.1$ and $\triangle t = 0.001$ because of the computing problems with smaller $\triangle x$ increments.

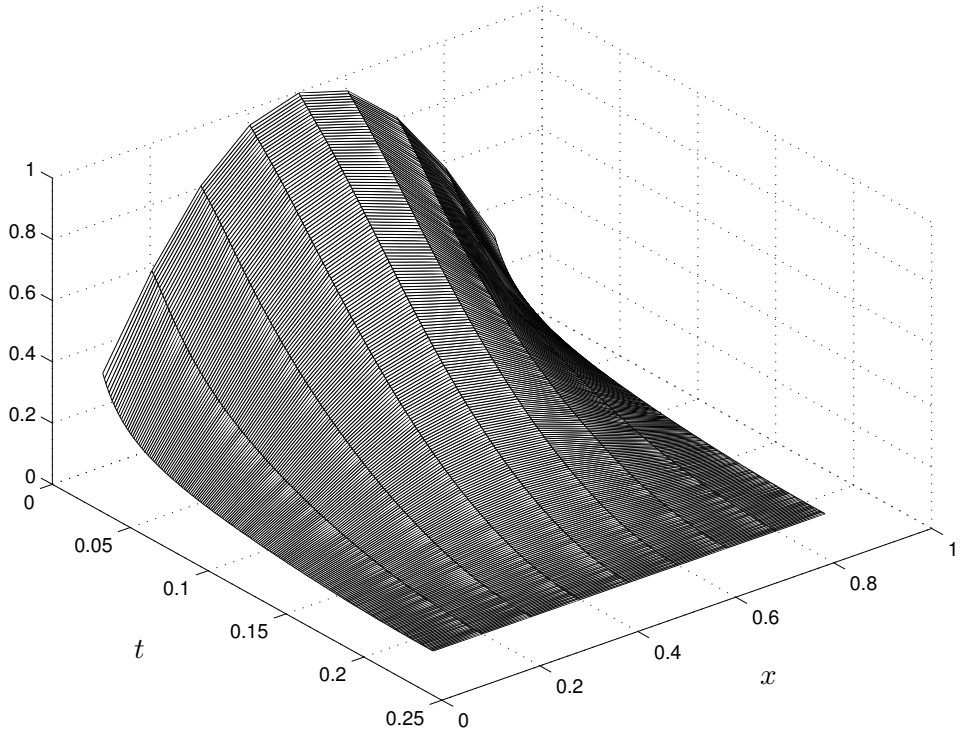

Figure 4.2. Solution to equation (1.23) with $\Phi = \sqrt{u}$ and $F = u^3$. $\triangle x = 0.1$ and $\triangle t = 0.001$.

We see that by making q bigger and keeping p the same, we cause extinction to occur at a slightly later time, but only very slightly. In our next two examples, we will look at what happens when we keep q constant, and increase p. We will choose $\Phi = u$ and $F = \sqrt{u}$, and $\Phi = u^3$ and $F = \sqrt{u}$. Figure 4.3 shows that extinction occurs at $t = 0.468$ for the first case, and for the second case, Figure 4.4 shows extinction of the solution occurs at $t = 1.305$. For these examples we have chosen $\triangle x = 0.05$ and $\triangle t = 0.001$ because this makes our stability condition $\Phi'(u) \leq 1.25$. This condition is always true for cases where $\Phi = u^p$ with $p \geq 1$, along with our initial condition.

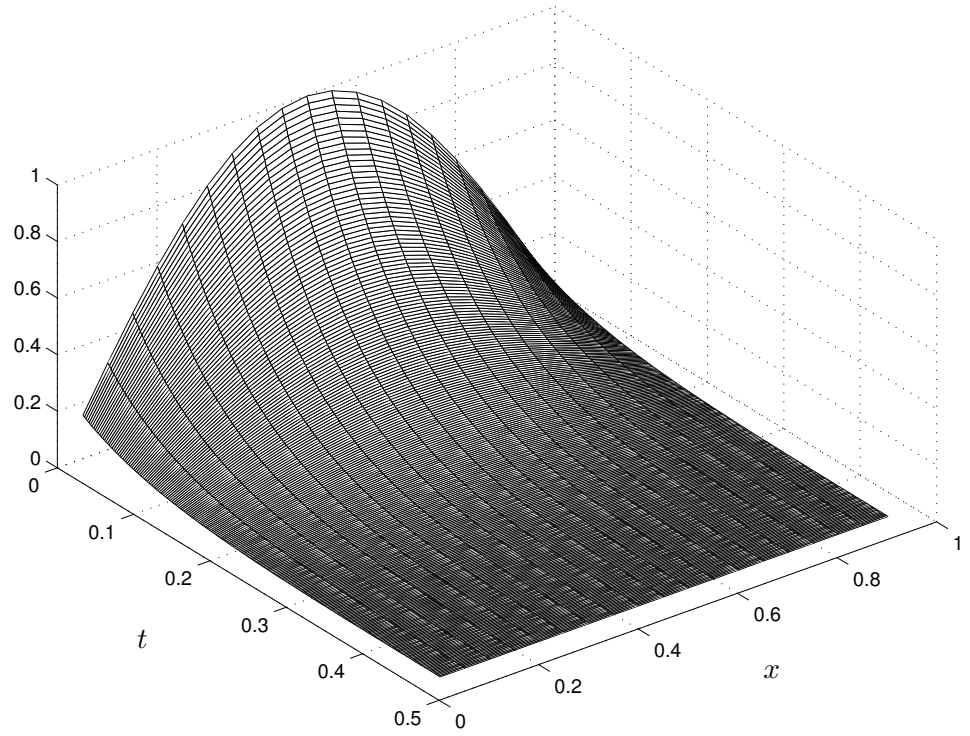

Figure 4.3. Solution to equation (1.23) with $\Phi = u$ and $F = \sqrt{u}$. $\triangle x = 0.05$ and $\triangle t = 0.001$.

This means that, as before, raising the power of one variable while the other remains constant will increase the extinction time of the solution. Although the solution becomes extinct at a later time, it still becomes extinct as it is supposed to. We see here that Φ seems to have a much greater influence on the solution than does F.

So, if increasing the power of Φ or F in (4.2) increases extinction time, then making both p and q less than 1 should cause extinction to occur at an earlier time. From Figure 4.5 we see that with $\Phi = \sqrt{u}$ and $F = \sqrt{u}$ we get an extinction time of $t = 0.192$.

We note here that with p and q both greater than 1, we do not have extinction until infinity. Although we do not show any plots of situations such as this, the code will run until the solution becomes so small that the computer rounds the solution to zero. The program that was used to run this code starts to round to zero at 10^{-324}.

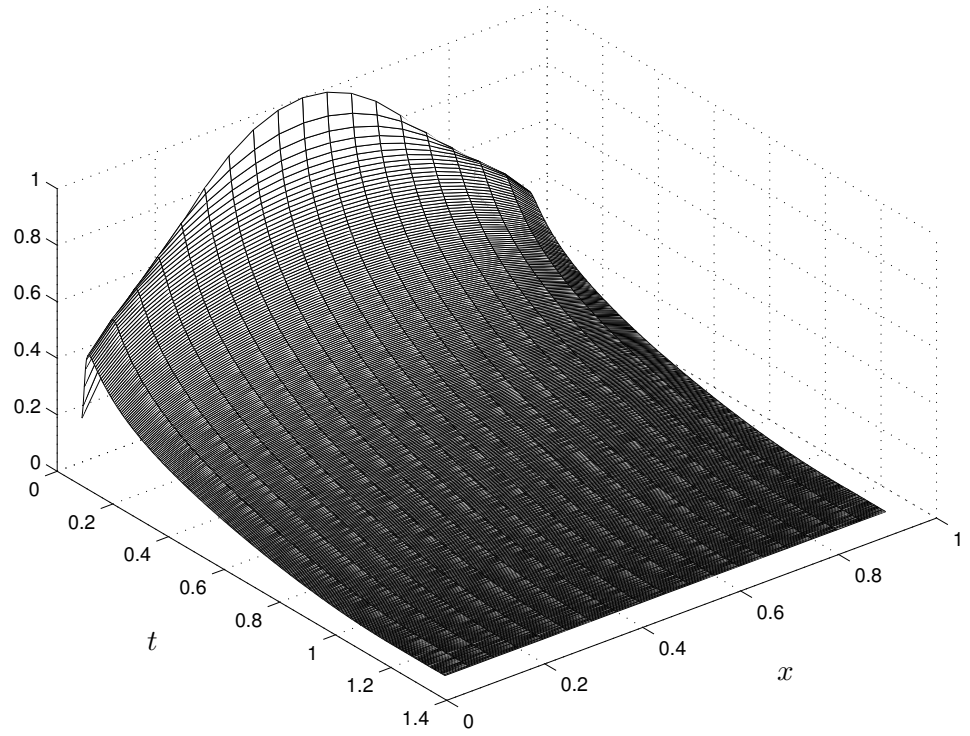

Figure 4.4. Solution to equation (1.23) with $\Phi = u^3$ and $F = \sqrt{u}$. $\triangle x = 0.05$ and $\triangle t = 0.001$.

This is not promising. The fact that the program will eventually round the solution to zero, makes it hard to tell whether extinction occurs at infinity or at a finite time. This especially comes into play when we have a Φ and F that have a large finite extinction time.

As examples of solutions with infinite extinction times, we choose $\Phi = u$ and $F = 0$, and $\Phi = u$ and $F = u$ because their solutions reach 10^{-324} in the least number of iterations and therefore, they have the smallest t for which the program would round to a zero solution of all the solutions that become extinct at infinity. The solutions to these particular problems are rounded to zero at just over $t = 100$.

On a more promising note, we have observed that when finite extinction time is supposed to occur, the observed extinction times are very small compared to $t = 100$. In a previous example, our numerical solution with $\Phi = u^3$ and $F = \sqrt{u}$ goes to zero at 10^{-6}. This means that for the iteration just before the numerical solution goes to zero,

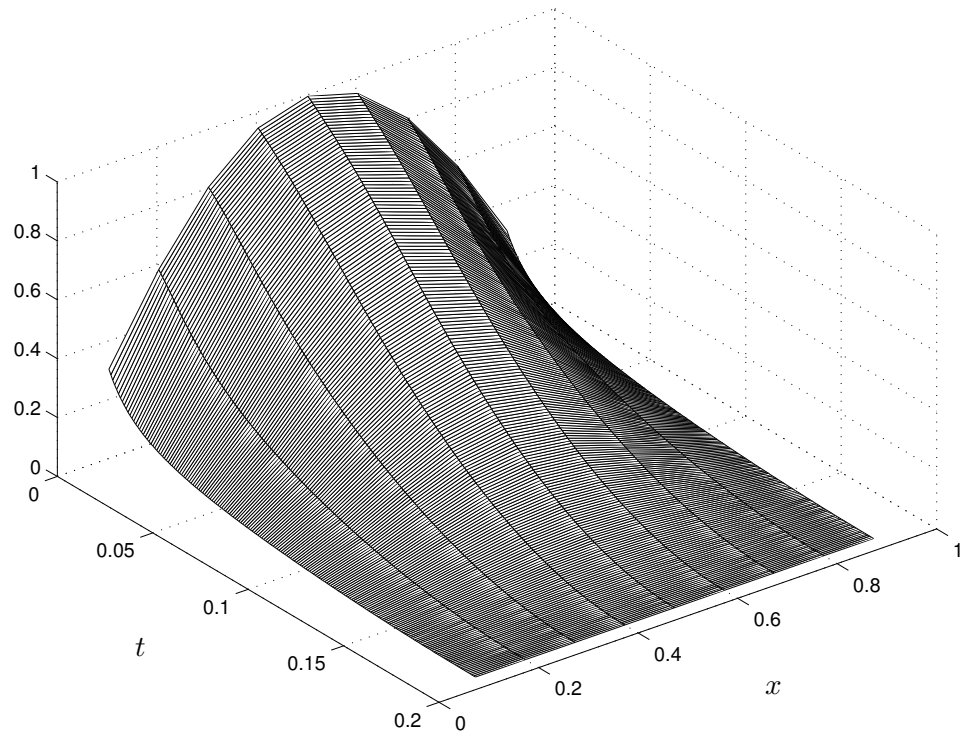

Figure 4.5. Solution to equation (1.23) with $\Phi = \sqrt{u}$ and $F = \sqrt{u}$. $\triangle x = 0.1$ and $\triangle t = 0.001$.

the numerical solution is on the order of 10^{-6}. So, the numerical solution goes to zero at a much larger solution than 10^{-324}.

Obviously we would like to test a case with a large finite extinction time such as $\Phi = u^{100}$ and $F = u^{0.999}$ to see what happens. The problem with this is that the code cannot handle such a case. The aforementioned case creates ill-conditioned matrices for which a solution cannot be found. The code can handle $\Phi = u^4$ and $F = u^{3/4}$. From Figure 4.6 we see that we have an extinction time at $t = 3.390$. Also, the iteration just before the numerical solution goes to zero is on the order of 10^{-12}. Since the code can only handle a limited number of cases, and the limits of these cases have extinction times far less than $t = 100$, then we feel justified in saying that we can predict whether finite extinction time will occur for any given case that the code can handle.

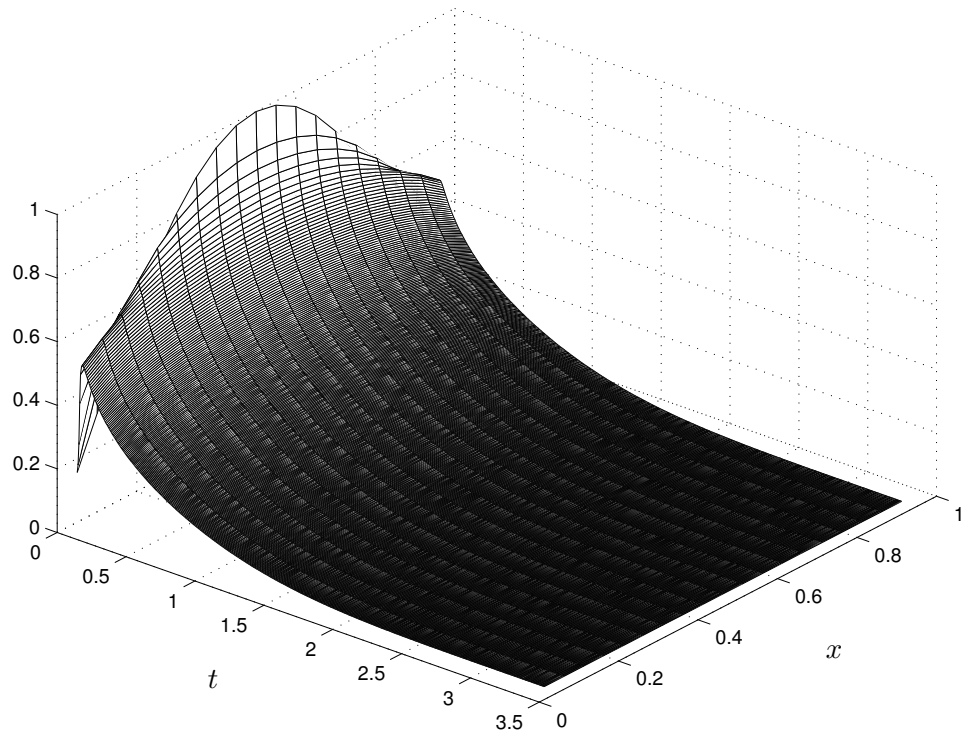

Figure 4.6. Solution to equation (1.23) with $\Phi = u^4$ and $F = u^{3/4}$. $\triangle x = 0.05$ and $\triangle t = 0.001$.

The code can handle any $\Phi = u^p$, where $p \geq \frac{1}{2}$. It can also handle any $F = u^q$, where $q \geq 0$. The only exception to this is if the difference between p and q is large. Also, remember that the stability condition must be satisfied. As a final example, Figure 4.7 shows $\Phi = u^{3/4}$ and $F = u^4$. This case has a finite extinction time at $t = 0.413$.

4.3 Comparison of the absorption-diffusion problem and the diffusion-free problem

In this section we will compare the numerical solution of the one-dimensional version of equation (1.23) and the numerical solution of (1.24) with $M = 1$. Equation (1.23) has the same boundary and initial conditions as in Section 4.2. Since we are concerned with whether the solution, $u(x,t)$, of (1.23) is less than the solution, z(t), of (1.24), then we will only compare the $\max_i\{u(x_i,t)\}$ to $z(t)$, at each t. In this thesis, we have noticed that for the equations and initial conditions considered, the $\max_i\{u(x_i,t)\} = u(0.5,t)$.

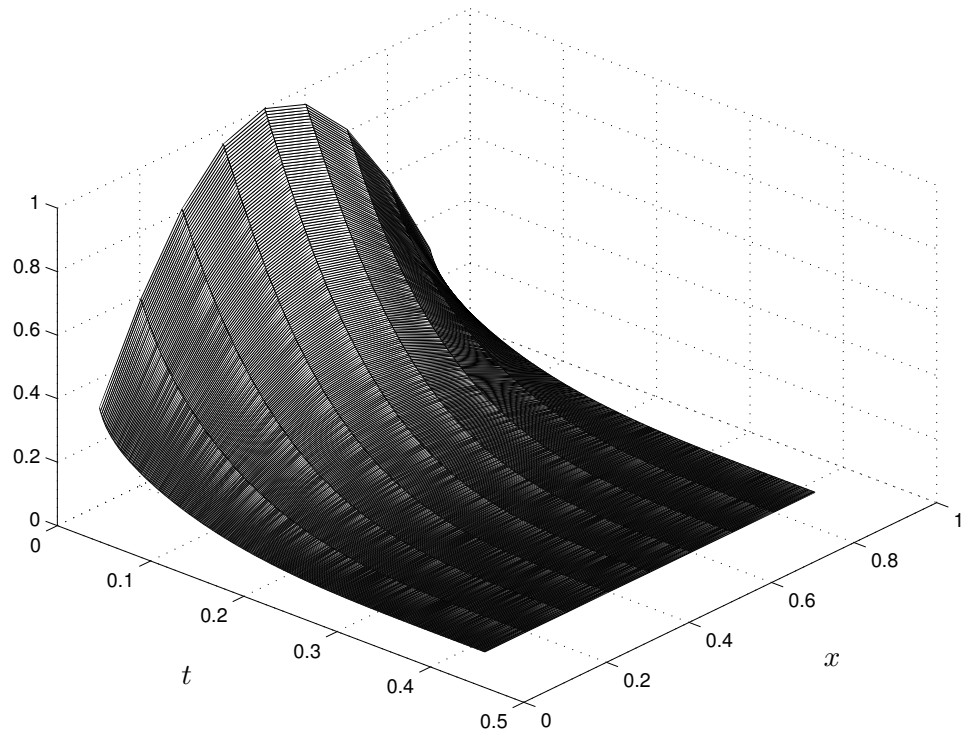

Figure 4.7. Solution to equation (1.23) with $\Phi = u^{3/4}$ and $F = u^4$. $\triangle x = 0.1$ and $\triangle t = 0.001$.

Since equation (1.24) has finite extinction time if and only if equation (1.25) holds, we will look at cases where Φ and F satisfy (1.25) first. For the sake of comparison, we will look at the same cases as we did in Section 4.2.

We notice that in Figures 4.8, 4.9, 4.10, 4.11, 4.12, 4.13, and 4.14, $u \leq z$. This is exactly what we wanted, and is a good sign that the solution of (1.24) becomes extinct in finite time when (1.25) holds.

Now, we turn to cases where there is no finite extinction time. We will look at $\Phi = u$ and $F = u$, and $\Phi = u^3$ and $F = u^3$. Figure 4.15 shows the former. Although we only show the first 3 time units, the numerical solution does not get rounded to zero until after $t = 100$, which was our full run-time. We found that $u \leq z$ for all t.

Turning to the case where $\Phi = u^3$ and $F = u^3$, we clearly see from Figure 4.16 that $u \leq z$. This solution is rounded to zero well after our run-time of $t = 100$, and $u \leq z$ for

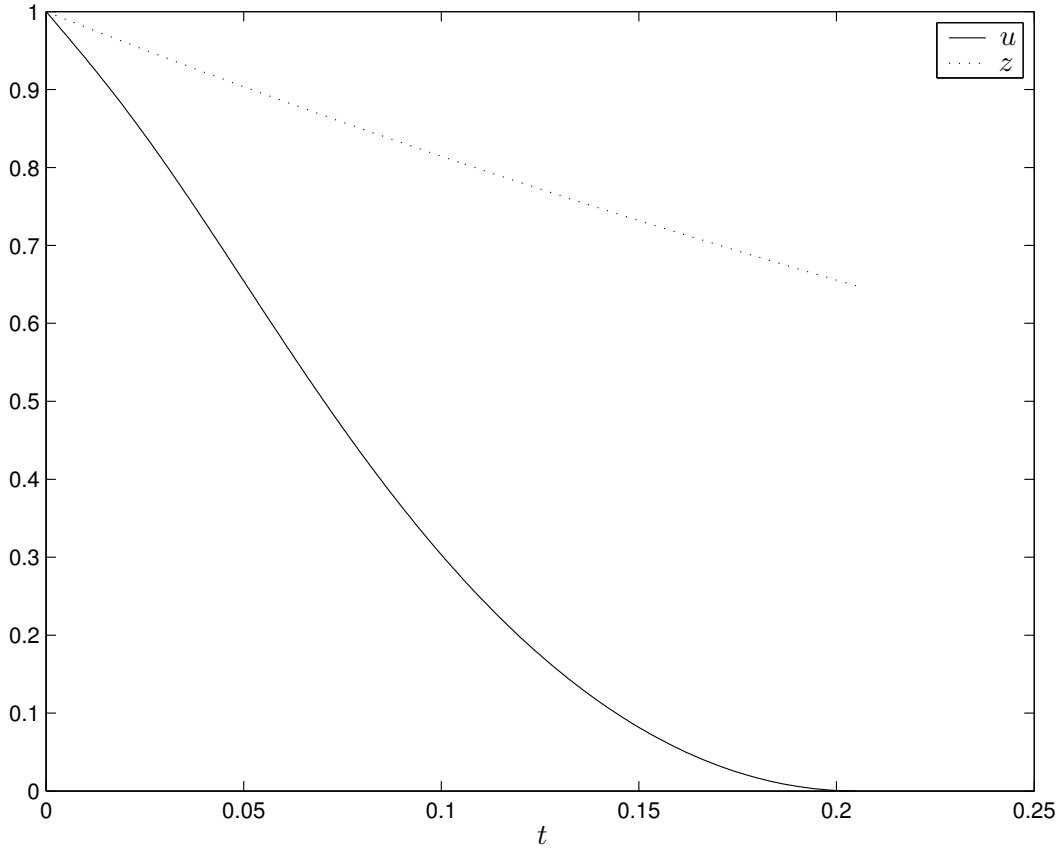

Figure 4.8. Comparison of solutions of equation (1.23) and (1.24) with $\Phi = \sqrt{u}$ and $F = u$. $\triangle x = .1$ and $\triangle t = .001$.

the entire time. This means that we have found $u \leq z$ for every case that we tested, no matter when the extinction time. This result is even better then we expected, and is a good sign that $u \leq z$ for all Φ and F with the same boundary and initial conditions.

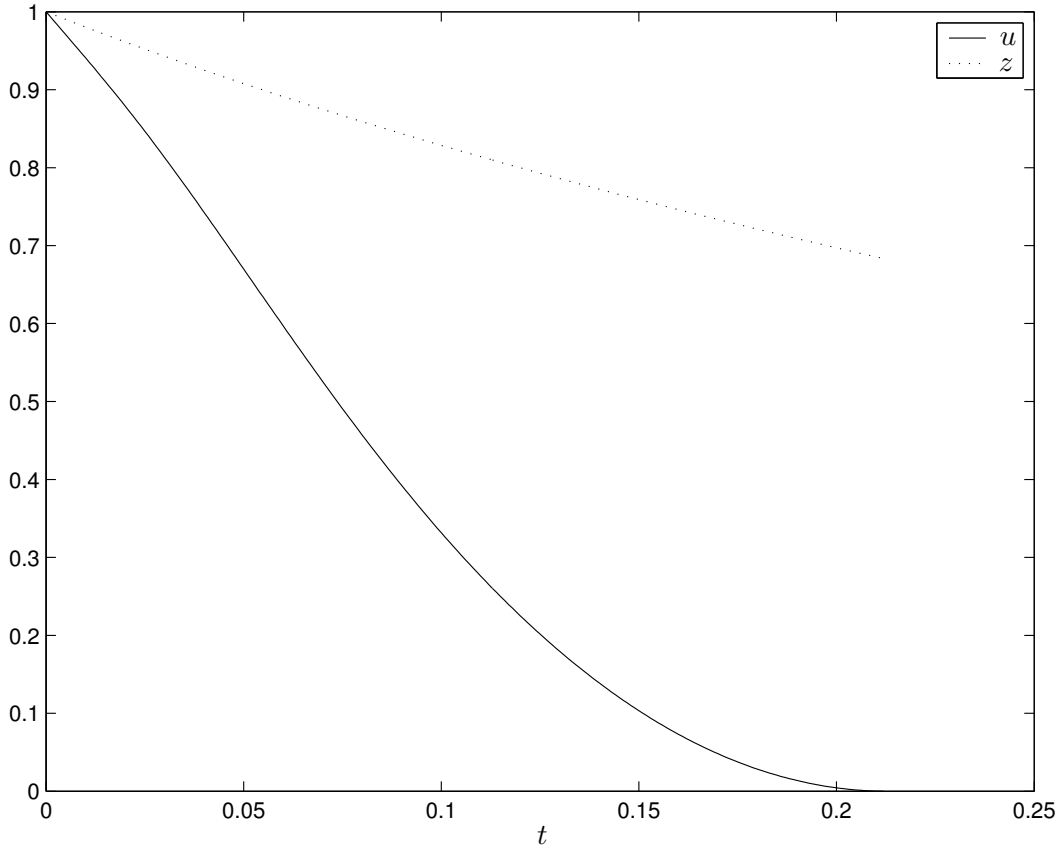

Figure 4.9. Comparison of solutions of equation (1.23) and (1.24) with $\Phi = \sqrt{u}$ and $F = u^3$. $\triangle x = 0.1$ and $\triangle t = 0.001$.

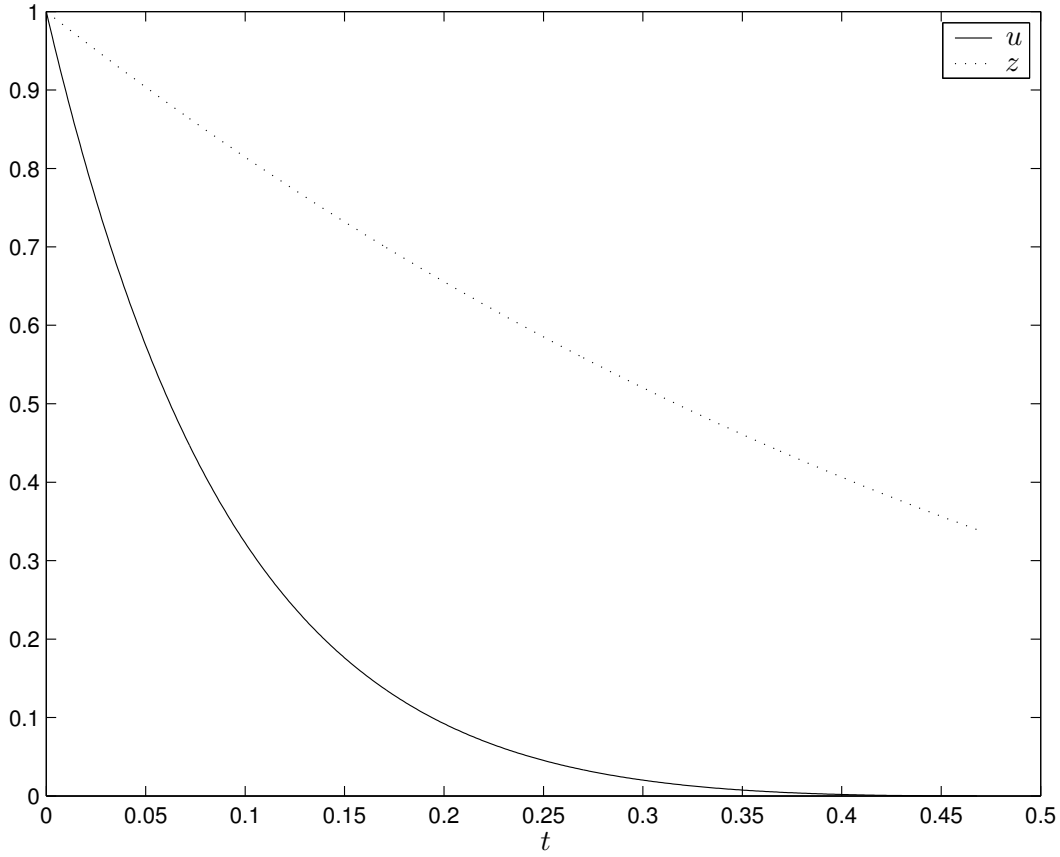

Figure 4.10. Comparison of solutions of equation (1.23) and (1.24) with $\Phi = u$ and $F = \sqrt{u}$. $\triangle x = 0.05$ and $\triangle t = 0.001$.

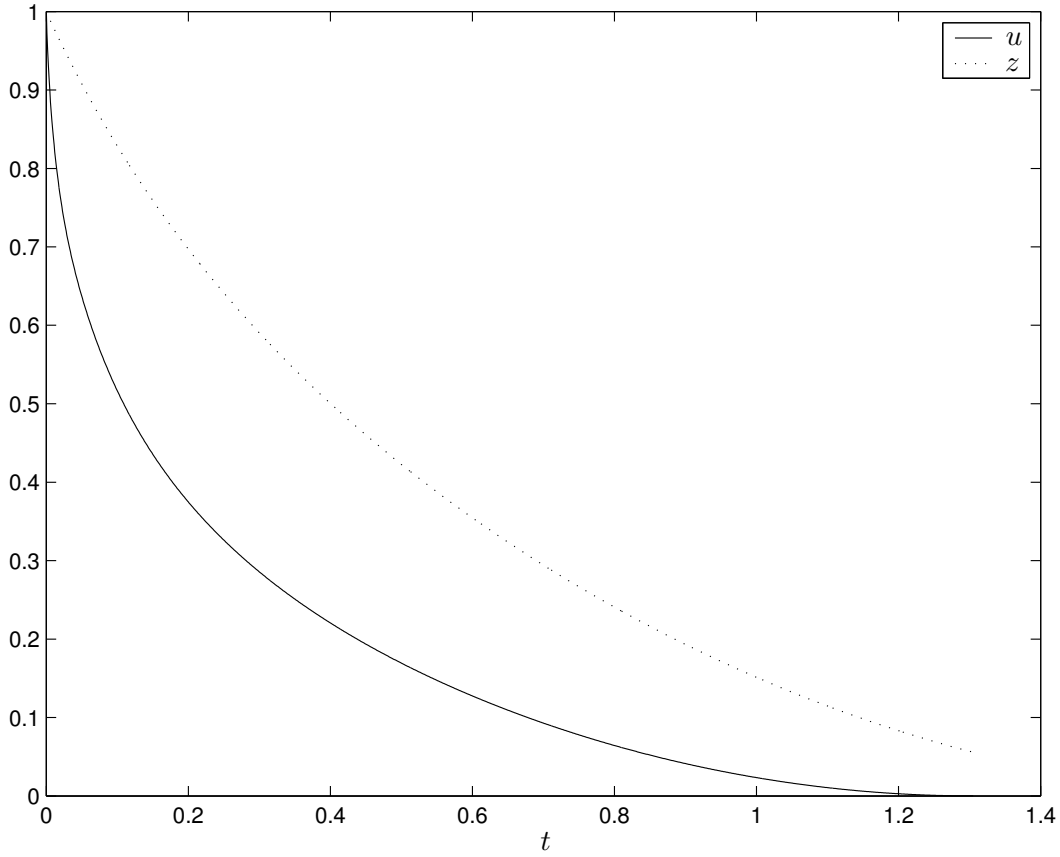

Figure 4.11. Comparison of solutions of equation (1.23) and (1.24) with $\Phi = u^3$ and $F = \sqrt{u}$. $\triangle x = 0.05$ and $\triangle t = 0.001$.

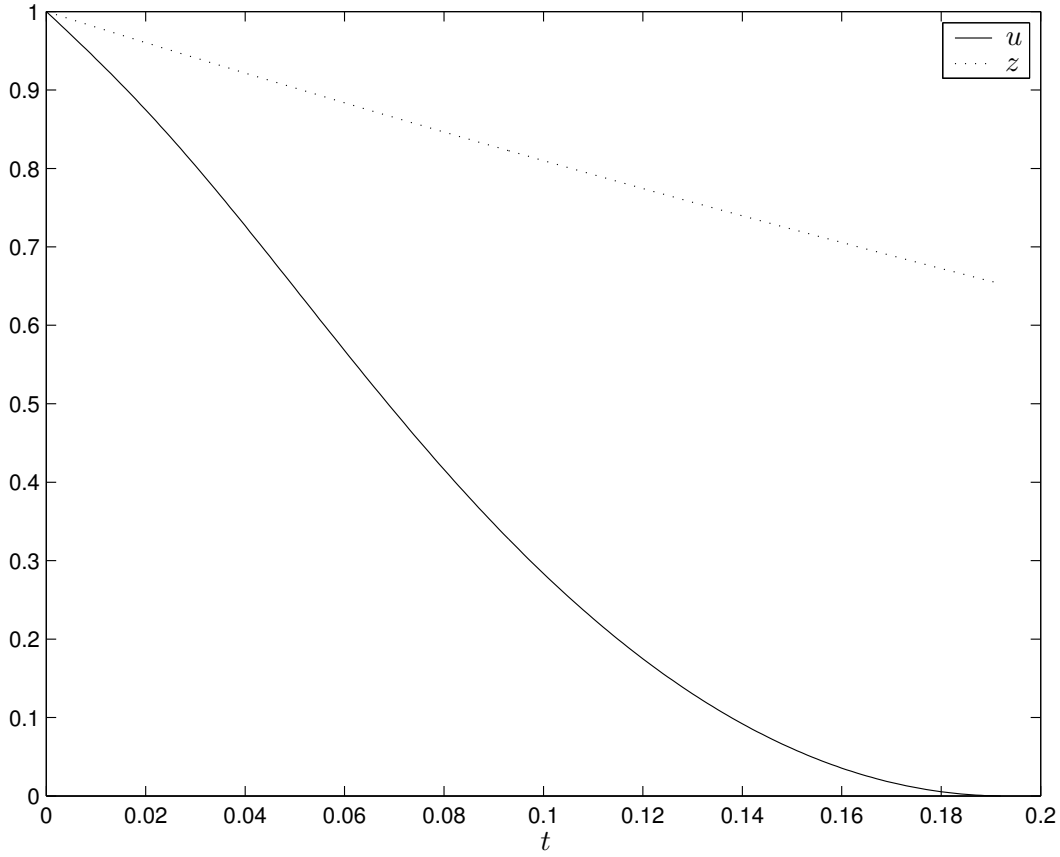

Figure 4.12. Comparison of solutions of equation (1.23) and (1.24) with $\Phi = \sqrt{u}$ and $F = \sqrt{u}$. $\triangle x = 0.1$ and $\triangle t = 0.001$.

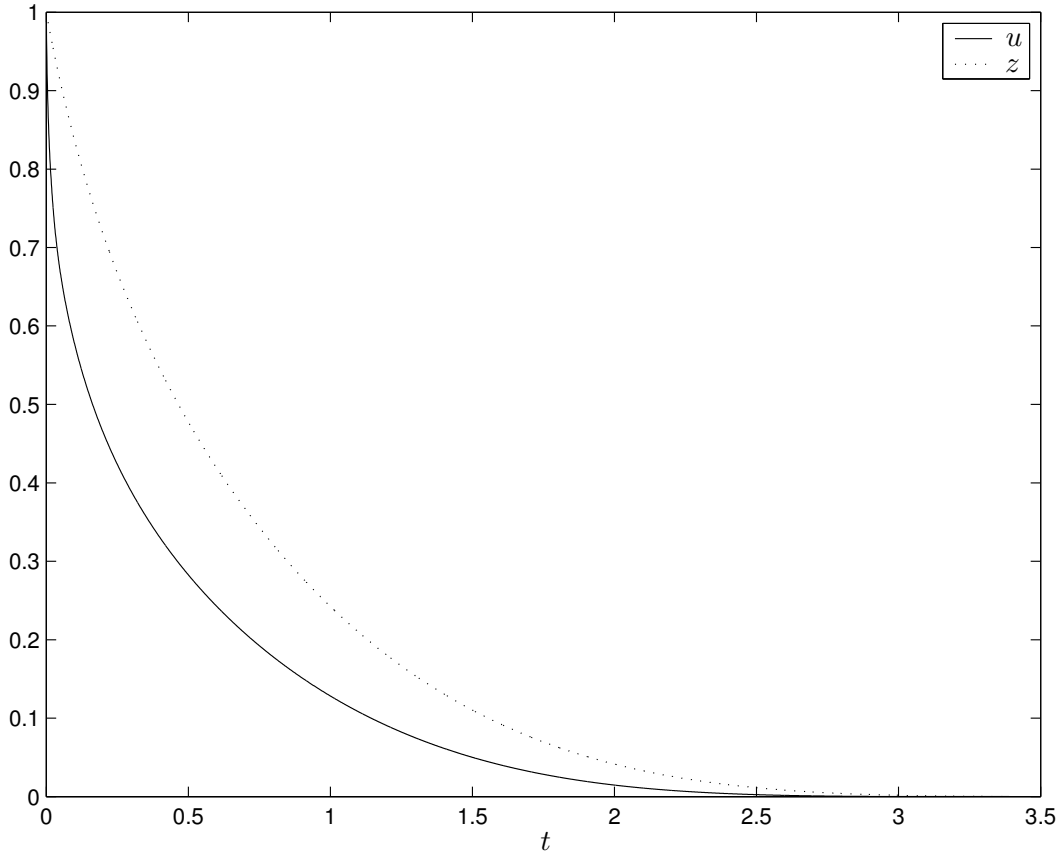

Figure 4.13. Comparison of solutions of equation (1.23) and (1.24) with $\Phi = u^4$ and $F = u^{3/4}$. $\triangle x = 0.05$ and $\triangle t = 0.001$.

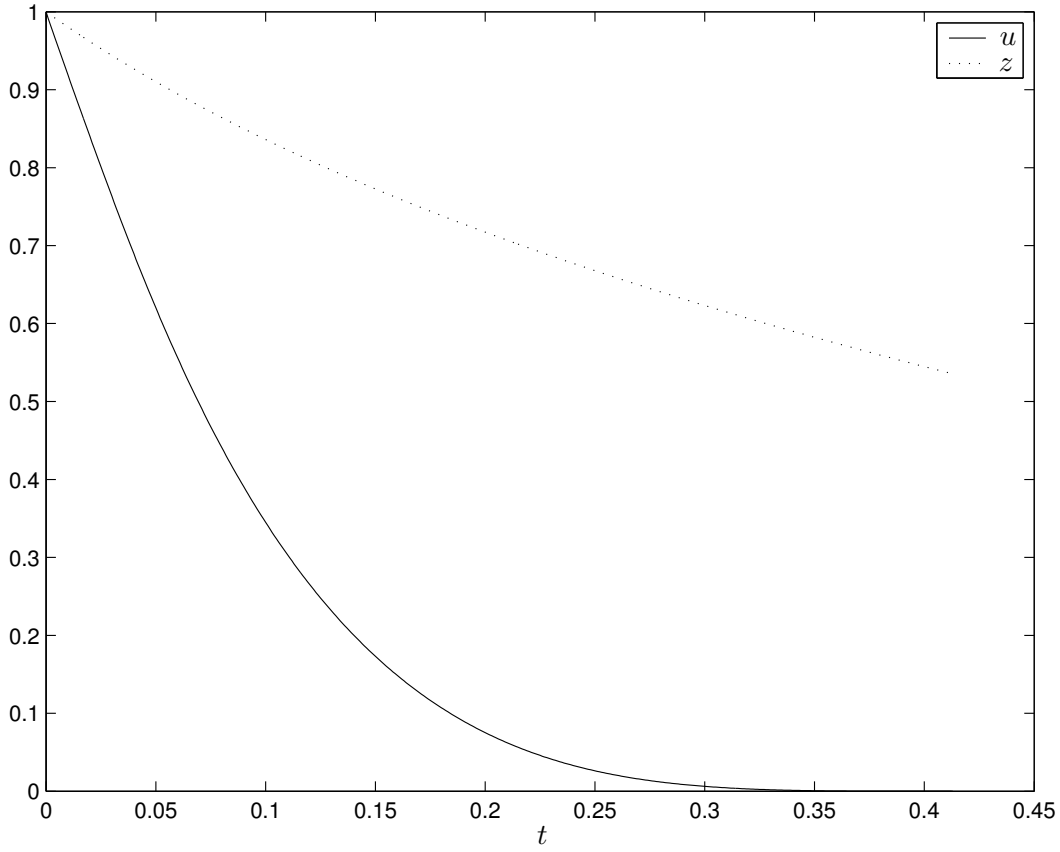

Figure 4.14. Comparison of solutions of equation (1.23) and (1.24) with $\Phi = u^{3/4}$ and $F = u^4$. $\triangle x = 0.1$ and $\triangle t = 0.001$.

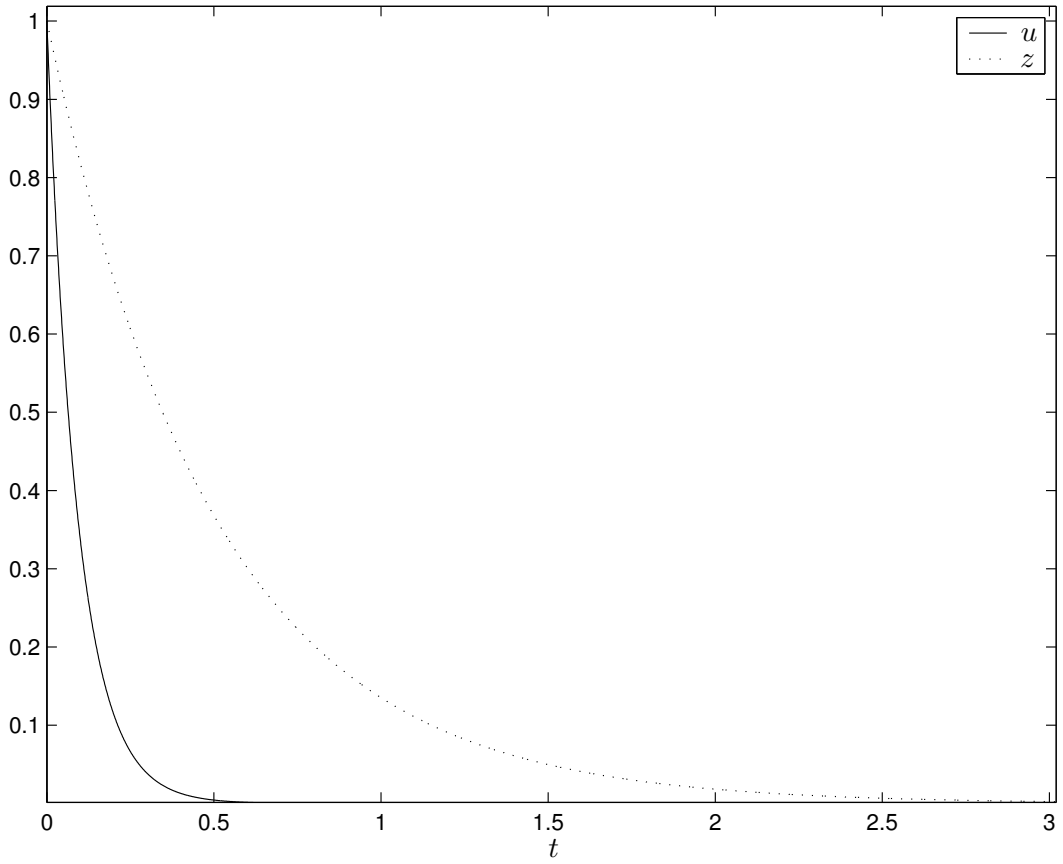

Figure 4.15. Comparison of solutions of equation (1.23) and (1.24) with $\Phi = u$ and $F = u$. $\triangle x = 0.05$ and $\triangle t = 0.001$.

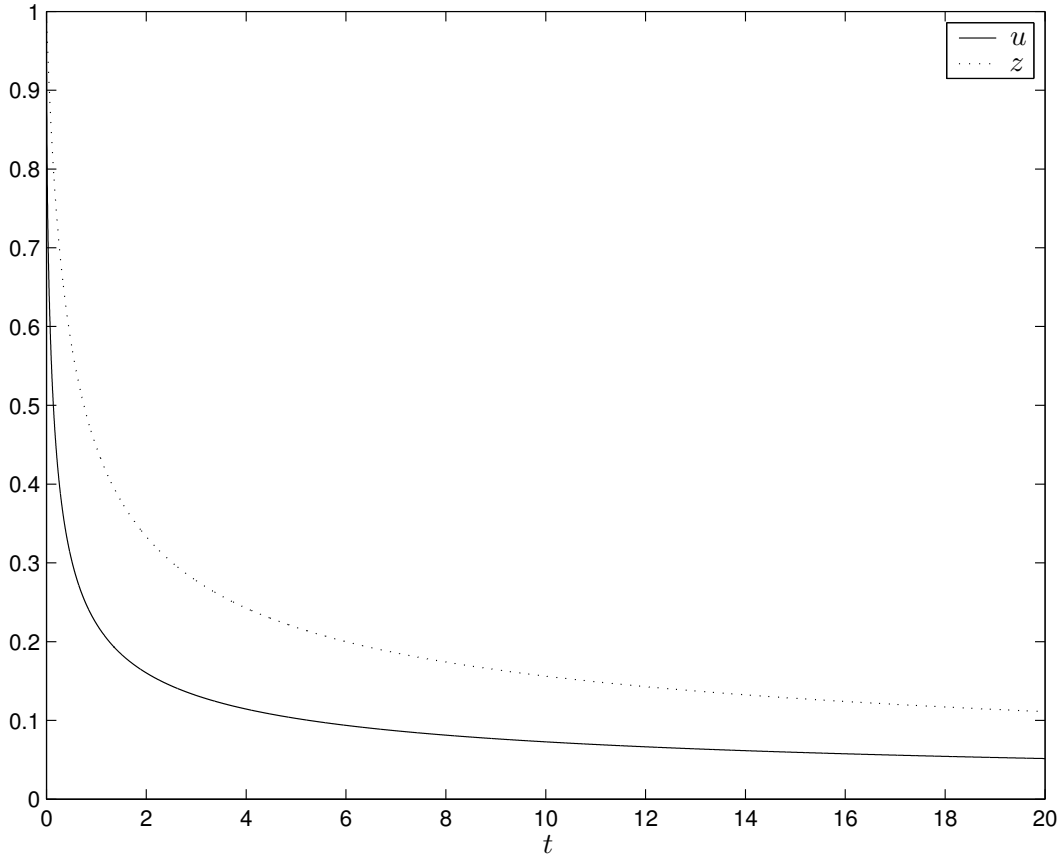

Figure 4.16. Comparison of solutions of equation (1.23) and (1.24) with $\Phi = u^3$ and $F = u^3$. $\triangle x = 0.05$ and $\triangle t = 0.001$.

V. Conclusion

5.1 Summary

In this thesis we have investigated the motivation for the n-dimensional absorption-diffusion equation. In doing so, we mathematically developed the one-dimensional, non-linear absorption-diffusion equation. We also proved analytically that for every absorption-diffusion equation with Φ and F satisfying certain conditions, extinction will always occur as $t \to \infty$.

Next, we looked at various examples of related problems. With heat pipe problems, we observed the development of a computational method based on finite-difference approximations. We then considered the dead core problem, which is a real world example of finite extinction time. Finally, we reviewed previous work that was done on finite extinction time problems for parabolic equations.

We then developed a numerical method in order to calculate solutions to the one-dimensional, non-linear absorption-diffusion equation. We tested the accuracy of this method on a one-dimensional, linear diffusion equation with a solution which could be written in closed form, and we saw that the numerical solution generated only a moderate amount of error. With such a small amount of error in our method, we were confident that our numerical scheme was accurate and proceeded to test various types of diffusion and absorption terms in order to see which ones produced extinction in finite time. Our extinction results were consistent with the analytical results of Lair and Oxley [10].

Next, we developed a numerical method to computationally solve the diffusion-free equation. This method was developed in a similar manner to the method for the one-dimensional, non-linear absorption-diffusion equation. This ensured that we had a high degree of accuracy and that we could compare the numerical solutions of the two. After comparison of the numerical solutions with our particular boundary and initial conditions, we found that for all of the cases that we tested and no matter when the extinction time, $u \leq z$.

We feel confident that given a Φ and F for which we can carry out our computational method, we can predict whether or not finite extinction time occurs. Although this sounds

promising, our computational method is limited in scope. For this reason, we were unable to address the open problem where we have slow diffusion and weak absorption, yet together they are strong.

The only known example that we have found that satisfies (2.6) is where Φ and F are piecewise linear functions which cross each other infinitely many times as they approach zero. Therein lies the problem, in that a computer cannot process infinitely many pieces.

5.2 Future Work

In the future, we would like to increase the order of accuracy of our approximations of u_t, u_x, and u_{xx} in order to increase the accuracy of our numerical solutions. We would also like to look at more comparisons between u and z in order to further justify that $u \leq z$ in all cases where we have finite extinction time, and even possibly when there is infinite extinction time.

We would also like to modify our method so that we can better handle cases with fast diffusion, i.e., cases where Φ' and Φ'' do not exist at the boundary. By doing this, we would hope that we could make our $\triangle x$ increments quite a bit smaller and still be able to obtain a numerical solution, which would be more accurate.

Finally, finding a different example of Φ and F which satisfy (2.6) and are useable in our code would be quite helpful in providing insight into the open problem. Even more useful would be an analytical proof of whether such Φ and F cause finite time extinction. If we cannot find a useable Φ and F, it would be beneficial to create a new method that can handle more cases with finite extinction time, in order to do a more in-depth study of fast diffusion and strong absorption conditions.

Appendix A. Matlab 6.1 Code

A.1 Absorption-Diffusion Code

```
function[z,i]=heateqn(phi,fu,delx,delt,t)

format long

syms u
phiprime=diff(phi,u);
phidblprime=diff(phi,u,2);

Nx=[delx:delx:1-delx]';
indi=(1/delx)-1;
indn=t/delt;
z(:,1)=sin(pi*Nx);

ppman=0;
pdpman=0;
fman=0;

if findsym(phiprime)==''
        phiprime=double(phiprime);
        ppinit=phiprime*ones(indi,indn+1);
        pp=phiprime*ones(indi,indn+1);
        ppman=1;
end
if findsym(phidblprime)==''
        phidblprime=double(phidblprime);
        pdpinit=phidblprime*ones(indi,indn+1);
        pdp=phidblprime*ones(indi,indn+1);
        pdpman=1;
end
if isa(fu,'double')==1
        funp=fu*ones(indi,indn+1);
        fman=1;
end

hwin=waitbar(0,'Please wait...');
for i=1:indn
        if ppman==0
```

```
        ppinit(:,i)=subs(phiprime,'u',z(:,i));
end
if pdpman==0
        pdpinit(:,i)=subs(phidblprime,'u',z(:,i));
end

ziplus1(1:indi-1,i)=z(2:indi,i);
ziplus1(indi,i)=0;
ziminus1(1,i)=0;
ziminus1(2:indi,i)=z(1:indi-1,i);

g=1-((2*delt)/(delx^2))*ppinit(:,i);
h=(delt/(delx^2))*ppinit(:,i)+(delt/(4*delx^2))*pdpinit(:,i).*ziplus1(:,i)- ...
        (delt/(4*delx^2))*pdpinit(:,i).*ziminus1(:,i);
p=(delt/(delx^2))*ppinit(:,i)+(delt/(4*delx^2))*pdpinit(:,i).*ziminus1(:,i)- ...
        (delt/(4*delx^2))*pdpinit(:,i).*ziplus1(:,i);

Q=diag(g);
Q=Q+diag(h(1:indi-1),1);
Q=Q+diag(p(2:indi),-1);

if fman==0
        funp(:,i)=subs(fu,'u',z(:,i));
end

z(:,i+1)=Q*z(:,i)-delt*funp(:,i);

[row,col]=find(z(:,i+1)<0);
if length(row)>=1
        z(row,i+1)=0;
end

[row,col]=find(z(:,i+1)==0);
if length(row)==indi
        i
        break
end

if ppman==0
        pp(:,i+1)=subs(phiprime,'u',z(:,i+1));
end
```

```
if pdpman==0
      pdp(:,i+1)=subs(phidblprime,'u',z(:,i+1));
end

zp(1:indi−1,i+1)=z(2:indi,i+1);
zp(indi,i+1)=0;
zm(1,i+1)=0;
zm(2:indi,i+1)=z(1:indi−1,i+1);

a=1+(delt/(delx^2))*pp(:,i+1);
b=(−delt/(2*delx^2))*pp(:,i+1)−(delt/(8*delx^2))*pdp(:,i+1).*zp(:,i+1)+ ...
      (delt/(8*delx^2))*pdp(:,i+1).*zm(:,i+1);
c=(−delt/(2*delx^2))*pp(:,i+1)−(delt/(8*delx^2))*pdp(:,i+1).*zm(:,i+1)+ ...
      (delt/(8*delx^2))*pdp(:,i+1).*zp(:,i+1);

Tone=diag(a);
Tone=Tone+diag(b(1:indi−1),1);
Tone=Tone+diag(c(2:indi),−1);

if ppman==0
      pp(:,i)=subs(phiprime,'u',z(:,i));
end
if pdpman==0
      pdp(:,i)=subs(phidblprime,'u',z(:,i));
end

zp(1:indi−1,i)=z(2:indi,i);
zp(indi,i)=0;
zm(1,i)=0;
zm(2:indi,i)=z(1:indi−1,i);

d=1−(delt/(delx^2))*pp(:,i);
e=(delt/(2*delx^2))*pp(:,i)+(delt/(8*delx^2))*pdp(:,i).*zp(:,i)− ...
      (delt/(8*delx^2))*pdp(:,i).*zm(:,i);
f=(delt/(2*delx^2))*pp(:,i)+(delt/(8*delx^2))*pdp(:,i).*zm(:,i)− ...
      (delt/(8*delx^2))*pdp(:,i).*zp(:,i);

Ttwo=diag(d);
Ttwo=Ttwo+diag(e(1:indi−1),1);
Ttwo=Ttwo+diag(f(2:indi),−1);
```

```
    if fman==0
        funp(:,i+1)=subs(fu,'u',z(:,i+1));
    end

    Rev=inv(Tone);
    Per=Rev*Ttwo;
    fud=funp(:,i+1)+funp(:,i);
    z(:,i+1)=Per*z(:,i)-Rev*[(delt/2)*fud];

    [row,col]=find(z(:,i+1)<0);
    if length(row)>=1
        z(row,i+1)=0;
    end

    [row,col]=find(z(:,i+1)==0);
    if length(row)==indi
        i
        break
    end

    waitbar(i/indn,hwin)
end
close(hwin)
```

A.2 Diffusion-Free Code

```
function[z]=zeqn(phi,fu,delt,t)

format long

syms u

indn=t/delt;
z(1)=1;

h=waitbar(0,'Please wait...');
for i=1:indn
    z(i+1)=z(i)-delt*(subs(fu,'u',z(i))+subs(phi,'u',z(i)));

    z(i+1)=z(i)-(delt/2)*(subs(fu,'u',z(i))+subs(phi,'u',z(i)))- ...
        (delt/2)*(subs(fu,'u',z(i+1))+subs(phi,'u',z(i+1)));

    waitbar(i/indn,h)
```

```
end
close(h)
```

Bibliography

1. Atkinson, Kendall E. *An Introduction to Numerical Analysis—2nd ed.* New York: John Wiley and Sons, Inc., 1989.

2. Bandle, Catherine and Ivar Stakgold. "The formation of the dead core in a parabolic reaction-diffusion equation," *Transactions of the American Mathematical Society*, vol. 286, pp. 275–293, 1984.

3. Chang, Won Soon and Gene T. Colwell. "Mathematical modeling of the transient operating characteristics of a low-temperature heat pipe," *Numerical Heat Transfer*, vol. 8, pp. 169–186, 1985.

4. Crank, John. *The Mathematics of Diffusion—2nd ed.* New York: Oxford University Press, 1975.

5. Diaz, Gregorio and Ildefonso Diaz. "Finite extinction time for a class of non-linear parabolic equations," *Communications in Partial Differential Equations*, vol. 4, pp. 1213–1231, 1979.

6. Farlow, Stanley J. *Partial Differential Equations for Scientists and Engineers.* New York: John Wiley and Sons, Inc., 1982.

7. Kalashnikov, A. S. "The propagation of disturbances in problems of non-linear heat conduction with absorption," *USSR Computational Mathematics and Mathematical Physics*, vol. 14, pp. 70–85, 1974.

8. Kersner, Robert. "Nonlinear heat conduction with absorption: space localization and extinction in finite time," *SIAM Journal on Applied Mathematics*, vol. 43, no. 6, pp. 1274–1285, 1983.

9. Lair, Alan V. "Finite extinction time for solutions of nonlinear parabolic equations," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 21, no. 1, pp. 1–8, 1993.

10. Lair, Alan V. and Mark E. Oxley. "Extinction in finite time of solutions to nonlinear absorption-diffusion equations," *Journal of Mathematical Analysis and Applications*, vol. 182, no. 3, pp. 857–866, 1994.

11. Smith, Gordon Dennis. *Numerical Solution of Partial Differential Equations: Finite Difference Methods—2nd ed.* Oxford: Oxford University Press, 1978.

12. Young, David M. and Robert Todd Gregory. *A Survey of Numerical Mathematics.* New York: Dover Publications, Inc., 1988.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 074-0188

The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of the collection of information, including suggestions for reducing this burden to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to an penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From – To)*
09-11-2002	**Master's Thesis**	Jun 2001 – Sep 2002

4. TITLE AND SUBTITLE

FINITE EXTINCTION TIME FOR NON-LINEAR ABSORPTION-DIFFUSION EQUATIONS

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

Goldberg, Jacob B.

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(S)

Air Force Institute of Technology
Graduate School of Engineering and Management (AFIT/EN)
2950 P Street, Building 640
WPAFB OH 45433-7765

8. PERFORMING ORGANIZATION REPORT NUMBER

AFIT/GAM/ENC/02S-02

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

N/A

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

In this thesis, we develop a numerical method in order to approximate the solutions of one-dimensional, non-linear absorption-diffusion equations. We test our method for accuracy against a linear diffusion equation with a solution that can be written in closed form. We then test various types of diffusion and absorption terms to determine which ones produce extinction in finite time.

We also develop a numerical method to computationally solve diffusion-free equations. We compare the numerical solutions of the one-dimensional, non-linear absorption-diffusion equation and the diffusion-free equation and we find that for the cases tested, the numerical absorption-diffusion solutions are always less than the numerical diffusion-free solutions. Furthermore, we find this is true for the cases tested when there is finite and infinite extinction time.

We also look at the open problem where we have slow diffusion and weak absorption but, their combined effect is strong. Our results provide some insight into the answer of this problem.

15. SUBJECT TERMS

Finite Extinction Time, Non-Linear, Absorption-Diffusion, Parabolic

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			Alan V. Lair (ENC)
U	**U**	**U**	**UU**	70	**19b. TELEPHONE NUMBER** (937) 255-3636, ext 4519; e-mail: Alan.Lair@afit.edu